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Application of differential dynamic programming to nuclear fuel management optimization

Larry Edgar Fennern
Iowa State University

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by

Larry Edgar Fennern

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NOMENCLATURE

Symbols	Description	Units
English		
$a_{r,s}$, $a^{T}_{r,s}$	Elements of burn matrix and the transpose of the burn matrix	sec^{-1}
$\underline{a}_k$	Constant vector in Equation (35)	
$A_{i_k,j}$	Burn matrix containing reaction rate data for depletion calculations for time step i_k and fuel batch j	sec^{-1}
$\hat{A}_{i_k,j}$	Matrix component of burn matrix, $A_{i_k,j}$, which contains neutron transmutation data	sec^{-1}
$\underline{b}$	Constant vector in Equations (36)	
$B_{i_k,j}$	Single step transmutation matrix for time step i_k and fuel batch j	
$c^{e}_{i,j}$	Cost contributions to the total performance index of fuel batch j incurred after an even stage i and prior to an odd stage i+1	mills/kwh
$c^{f}_{N,j}$	Final cost contributions after the fuel stage N to the total performance index of fuel batch j	mills/kwh
$c^{i}_{I,j}$	Initial cost contributions prior to stage I to the total performance index of fuel batch j	mills/kwh
$c^{o}_{i,j}$	Cost contributions to the total performance index of fuel batch j incurred after an odd stage i and prior to an even stage i+1	mills/kwh
$\underline{c}_k$	Constant vector in Equations (36)	

C_E	Cost of enriched uranium	\$
C_F	Cost of feed material per mass	\$/kgU
C_S	Cost of enrichment per separative work unit	\$/swu
d_k	Constant in Equations (36)	
D	Diagonal matrix	
$\underline{e}_k$	Constant vector in Equations (36)	
$\underline{e}_r$	Eigenvector	
E	Electrical energy produced per unit mass of uranium	kwh/kg
$f_{I,j}$	The function $c^i_{I,j} + c^f_{I,j}$	mills/kwh
$\underline{f}$	Constant vector in Equations (36)	
$\underline{f}_{k\,I,j}$	Column vector whose r^{th} component is $\frac{\partial}{\partial[\underline{n}_{I,j}]_r} f_{I,j}$	$\frac{\text{mills}}{\text{kwh}} \frac{\text{cm}^3}{\text{atoms}}$
F	Mass of feed material for a gaseous diffusion enrichment process	kgU
$F_d(t)$	Depreciation during the t^{th} month	mills/ month-kg
g_m	Constants in Equations (36)	
$\underline{g}_{i,j}$	State vector function	atoms/cm^3
$G_{x\,I,j}$	Matrix whose r^{th} column is the vector $\frac{\partial}{\partial[\underline{n}_{I,j}]_r} \underline{g}_{I,j}$	
$h_{I,j}$	Pseudo-Hamiltonian function defined by $f_{I,j} + \lambda^T \underline{g}_{I,j}$	mills/kwh

$\underline{h}_{x\,I,j}$	Column matrix whose r^{th} component is $\frac{\partial}{\partial[\underline{n}_{I,j}]_r} h_{I,j}$	$\frac{\text{mills}}{\text{kwh}} \frac{\text{cm}^3}{\text{atoms}}$
$H_{xx\,I,j}$	Matrix whose rs^{th} component is $\frac{\partial^2}{\partial[\underline{n}_{I,j}]_r \, \partial[\underline{n}_{I,j}]_s} h_{I,j}$	$\frac{\text{mills}}{\text{kwh}} \frac{\text{cm}^6}{\text{atoms}^2}$
I_f	Present worth of the salvage value referenced to the end of reactor operation	mills/kg
I_i	Sum of the present worth of all the depreciable investments referenced to the start of reactor operation	mills/kg
m	Length of reactor operating time period for the given fuel loading	months
$m_{r,s}$, $m^{-1}_{r,s}$	Elements of modal and inverse modal matrices	
M, M^{-1}	Modal and inverse modal matrices	
M	Money per unit mass of uranium	\$/kg
$\underline{n}$, $\underline{n}_{i,j}$	Vector of atom densities	atoms/cm^3
N	Total number of stages	
P	Mass of product material from a gaseous diffusion enrichment process	kgU
PI_I	Performance index summed from stage I to the final stage and summed over all fuel batches	mills/kwh
$PI_{I,j}$	Performance index summed from stage I to the final stage for fuel batch j	mills/kwh
$\underline{PI}_{x\,I,j}$	Column vector whose r^{th} component is $\frac{\partial}{\partial[\underline{n}_{I,j}]_r} PI_{I,j}\Big\|_{\underline{n}_{I,j} = \underline{n}^{+}_{I,j}}$	$\frac{\text{mills}}{\text{kwh}} \frac{\text{cm}^3}{\text{atoms}}$

$PI^{+}_{xx\,I,j}$	Symmetric matrix whose rs^{th} component is $\dfrac{\partial^2}{\partial[\underline{n}_{I,j}]_r\,\partial[\underline{n}_{I,j}]_s} PI_{I,j}\Big\|_{\underline{n}_{I,j} = \underline{n}^{+}_{I,j}}$	$\dfrac{\text{mills}}{\text{kwh}} \dfrac{\text{cm}^6}{\text{atoms}^2}$
q	Dimensions of vectors and matrices	
Q(t)	Quantity of electricity generated in t^{th} month	kwh/month-kg
r	Effective rate of return	
S	Separative duty	swu
$\underline{u}$, $\underline{u}_{i,j}$	Decision vector	
$\underline{u}_s$	Eigenvector of dual system	
x_f	Atom fraction of uranium-235 in the feed for a gaseous diffusion enrichment process	
x_p	Atom fraction of uranium-235 in the product of a gaseous diffusion enrichment process	
x_w	Atom fraction of uranium-235 in the tails of a gaseous diffusion enrichment process	
W	Mass of waste or tails material from a gaseous diffusion enrichment process	kgU
Z	Scaled burn matrix	sec^{-1}

Greek

ε	Error per stage	
λ_r	Eigenvalue	
$\Lambda_{i_k,j}$	Matrix component of burn matrix, $A_{i_k,j}$, which contains radioactive decay data	sec^{-1}

μ_s	Eigenvalue of dual system	
π_i	Set of decision vectors for all fuel batches prior to the i^{th} stage	
Π_i	Set of decision vectors for all fuel batches and stages i to the final stage	
$\sigma^g_{rs\ i_k,j}$	Microscopic cross section for energy group g and the rs^{th} element of $\hat{A}_{i_k,j}$	cm^2
τ	Income tax rate	
$\phi(x)$	Separation potential	swu/kgU
$\phi^g_{i_k,j}$	Neutron flux for energy group g and the burn matrix $A_{i_k,j}$	neutrons/ cm^2/sec
$\Phi_{i,j}$	State transition matrix for the i^{th} stage and j^{th} fuel batch	
ψ_i	Set of state vectors for all fuel batches prior to the i^{th} stage	atoms/cm^3
Ψ_i	Set of state vectors for all fuel batches and stages i to the final stage	atoms/cm^3

Notes

Superscripts

o	Means initial value
g	Means energy group
+	Means nominal state

Subscripts

j	Means fuel batch number
i_k	Means k^{th} time step of stage i
i	Means stage number
r,s	Means dimensions or matrix location

INTRODUCTION

Nuclear fuel management optimization has been defined in a variety of ways. Nuclear fuel management may refer to external or out-of-core management decisions, such as the purchase, conversion, enrichment, fabrication, transportation, and reprocessing of the nuclear fuel; or internal or in-core management decisions, such as the enrichment, placement and burnup of the nuclear fuel. The economy of the nuclear fuel cycle may be optimized with respect to an overall allocation of nuclear produced power from a number of reactors in the power system, to the refueling strategy for a number of fuel cycles during the lifetime of a given nuclear reactor, or to an initial loading and operating policy for a single fuel cycle for a given nuclear reactor.

The performance index or objective function to be optimized may include the fuel cycle costs, the ratio of critical mass to energy produced, or the burnup of the fuel. The independent variables used to optimize the performance index may include the fissile distribution of the fuel, the refueling rate, the power and flux distribution, the core dimensions, the placement, movement and discharge of the fuel, the cycle times and burnups, and the control material distribution. Generally, a number of constraints are imposed to assure a feasible optimal solution which may include criticality, reactivity, maximum burnup, critical heat flux, power peaking,

refueling volume, mechanical design and control constraints.

In this study the use of a nonlinear optimization technique, differential dynamic programming, to solve nuclear fuel management problems was considered. The performance index was defined as the levelized fuel cycle costs. Out-of-core components of this performance index were fixed, and an in-core optimization was considered. The optimization process was defined as determining the isotopic composition of the core regions of the reactor(s) which minimized the performance index. Constraints required for a solution to be feasible were criticality, maximum burnup, and gross power-peaking.

The method of differential dynamic programming was applied to the problem of optimizing the performance index for batch and zone loading with fresh or recycle fuel in a pressurized water reactor, for a refueling with fresh fuel and shuffling of partially burned fuel in a multi-zoned pressurized water reactor, and for a number of refuelings using recycle fuels in an interacting system consisting of a pressurized water reactor and a liquid metal fast breeder reactor. Numerical results for these studies are presented, and where practical, the accuracy and usefulness of the method are discussed. Specific recommendations are given concerning the practical application of differential dynamic programming to nuclear fuel management problems.

REVIEW OF LITERATURE

There have been a variety of approaches studied to optimize some aspect of the nuclear fuel management problem. Each approach has unique advantages and disadvantages which determine the practical usefulness of the method.

Initial fuel optimization studies focused on minimizing the critical mass. Goertzel [1] formulated a variational approach using an integral criticality equation to study this problem. Extensions of the variational approach to include piecewise continuous fuel distributions and two neutron energy groups were formulated by DeVooght [2] and Shapiro [3], respectively. Otuska [4] applied a perturbation technique to arrive at the same conclusions as Goertzel. The reactor models used in these studies were simple enough to permit an analytical solution.

Many investigators have attempted with some success to apply methods from the calculus of variations to minimize a more representative economic performance index. Goldschmidt and Quenon [5], using a one energy group slab diffusion reactor model, determined the fissile fuel distribution that minimized the ratio of the fissile mass to the energy produced. The application of the Maximum Principle of Pontryagin and the Multiplier Rule of Hestenes were used. Motoda [6] used a similar model and procedure to maximize the burnup for continuous scattered refueling.

Wade and Terney [7] chose a weighted performance index which included the energy released, burnup, peak central temperatures and critical heat flux. A quasistatic, modified one group theory neutron balance nodal equation was used to describe reactor physics, and a control sequence determined to minimize the performance index by an iterative procedure. A control sequence was chosen and the state equations and performance index were calculated forward in time. The Lagrange multipliers, introduced upon the application of Pontryagin's Maximum Principle as extended by Berkovitz, were then determined backwards in time, and a new control sequence determined. The procedure was repeated until a convergence criteria was satisfied. Gradient and linear programming was used to determine the new control vector.

Motoda [8] has also studied the control problem using the calculus of variations to achieve maximum average burnup in a one dimensional cylindrical light water reactor. An extension of these basic techniques to minimize the fuel cycle cost function by determining the optimal fuel enrichment distribution in fast reactors has been studied by Goldschmidt [9, 10]. The greatest difficulties in using the traditional methods from the calculus of variations to solve fuel management problems are the extreme complexity and computational impracticality that occur when reactor models with sufficient detail to satisfactorily model a large power reactor are introduced.

Linear and quadratic programming are two other optimization methods which have been used in fuel management studies. Tabak [11] minimized the mass of uranium-235 and maximized the mass of plutonium-239. Suzuki and Kiyose [12] minimized the stagewise consumption of fresh fuel by maximizing the burnup. Sauar [13] minimized the present worth of fuel cycle costs. The application of these techniques required considerable simplification of the reactor physics equations. Sauar used the simple linear model and a state vector with the single element, k_∞, to search out an optimal solution for fuel shuffling and loading distributions. A parametric variation of the remaining variables using the linear program as a subprogram yielded a global optimum. More detailed computations were used to check for feasibility and to correct the data.

Fuel scattering and loading models have also been studied by Fagan and Sesonske [14], who used a quaziequilibrium cycle and a direct search routine. Naft and Sesonske [15] used quarter core symmetry, shuffling rules, and dual exchanges to reduce the number of shuffling possibilities. Starting from a base point, the best dual exchanges as given by an incremental power peaking performance index were conducted sequentially until all the fuel had been shuffled. The procedure was repeated until no dual exchange was successful in minimizing the performance index. Essentially, these techniques fall into the class of integer programming.

Another approach to fuel management optimization which has promised some degree of success and usefulness is dynamic programming. Wall and Fenech [16] considered this approach in optimizing the loading strategy of a three zone pressurized water reactor over the lifetime of the plant. The method of dynamic programming hinges on Bellman's Principle of Optimality [17, 18] which states that at any stage of operation, only the current state of the system and future control actions determine optimality, not the previous history of the system. This principle reduces the number of unique loading possibilities which need to be considered. Unfortunately, the method requires large amounts of computer time and storage, which may be unfeasible, particularly if the state vector contains more than one element and the "curse of dimensionality" [17, 18] prohibits a solution.

Stover and Sesonske [19] used an improvement of the dynamic programming routine in order to optimize scatter loading in a boiling water reactor. A reduction in computation time and storage occurred by the use of the "Elimination of Similar End States" principle. For those end state variables which differed by less than a prescribed criteria, the most optimal end state was retained and the others were discarded. Civita, Fornaciari, and Mazzanti [20] used dynamic programming in looking at operation strategies such as stretch out of the cycle time.

Recently the method of approximate programming has been gaining attention in fuel management optimization studies. In this method the original nonlinear problem is reduced to a linear problem by expanding variables around a feasible solution and neglecting higher order terms. Starting with a promising feasible solution, a neighborhood search is conducted for a better solution. An iterative procedure follows. Motoda [21] applied this technique by descretizing the one-group one-dimensional diffusion equation, constraint relations and performance index by central and forward difference operations. Some further linearization was required to obtain a standard linear programming problem. Using a simplified state vector, the burnup of the fuel was maximized. Inoue [22] further applied this method to determine optimal design criteria for fast reactors, such as the number of fuel assemblies, core height, axial blanket length, fuel rod diameter, pitch of the core and blanket, and enrichments in the core at initial and equilibrium conditions, in order to minimize the fuel cycle costs.

Stoll and Axford [23] combined some of the previous ideas with a Lagrangian multiplier optimization technique in order to select enrichments, control poison, and fuel shuffling schemes for a fast reactor. Hence the Lagrange multiplier formulation was expressed in terms of small perturbations in the state variables. An Implicit Enumeration Method was used

by beginning with a feasible solution at a given cost and searching in a neighborhood of the state variables to find another feasible solution of lower cost.

Other fuel management optimization techniques have been developed. Mélice [24] plotted a "k-profile" of a pressurized water reactor core, and by various rules and procedures was able to synthesize a reloading pattern from this profile. Suzuki and Kiyose [25] used a method of topological mapping theory to determine a poison optimization scheme for a light water reactor in order to maximize burnup. A study using heuristic learning techniques for in-core fuel management was completed by Hoshino [26]. Heuristics techniques are rules, strategies, or tricks which limit the search for solutions, determined by a reinforcement-punishment learning technique.

APPLICATION OF DIFFERENTIAL DYNAMIC PROGRAMMING TO NUCLEAR FUEL MANAGEMENT PROBLEMS

Theory

Differential Dynamic Programming (DDP) is an iterative approximation technique for achieving optimal solutions of nonlinear systems [27]. It was developed from dynamic programming rather than from the calculus of variations; hence first derivatives do not appear as choice variables. This is an important consideration since the numerical solutions to the physical equations may make these derivatives difficult to determine. In DDP the performance index is expanded in terms of a truncated Taylor series about a set of state vectors, the trajectory, for a given set of control or decision vectors, the schedule. Expressions are then derived using the Principle of Optimality [17, 18] for the coefficients of the Taylor series. Once the Taylor series coefficients are determined, a search is conducted about a neighborhood of this nominal trajectory until a better trajectory is obtained, which then becomes the new nominal state. An iterative procedure follows which ends when an optimal solution is obtained. However, this solution is not necessarily a global optimum.

In the procedure, principles from dynamic programming are used to obtain the truncated Taylor series coefficients. It differs from dynamic programming in the extent of the search of the vector space defining the performance index, in the

storage of only the Taylor series coefficients, and in the iterative method of finding an optimal solution. The storage requirements are considerably less than dynamic programming. The computation time may be more or less for the DDP approach to fuel management problems depending on the number of iterations required to achieve an optimal solution. The reactor physics calculations are the most time consuming part of the computations. In DDP the effects of these computations are approximated in the Taylor series coefficients from the results of the nominal trajectory. No approximation is used in the dynamic programming approach, but rather an attempt is made to limit the number of possible trajectories which need to be considered.

The process of searching about a nominal trajectory for a better feasible solution resembles the method of approximate programming which has been applied to control rod programming and plant design [21, 22]. The procedure of obtaining coefficients for the expansion of the performance index is different, however, particularly since these studies did not consider any multi-stage processes.

In order to apply DDP to the nuclear fuel management problems, one can consider the loading, discharge, and reloading of a nuclear reactor as a multi-stage decision process as depicted in Figure 1. The q-dimensional state vector for the i^{th} stage and j^{th} fuel batch, $\underline{n}_{i,j}$, is defined as the atom

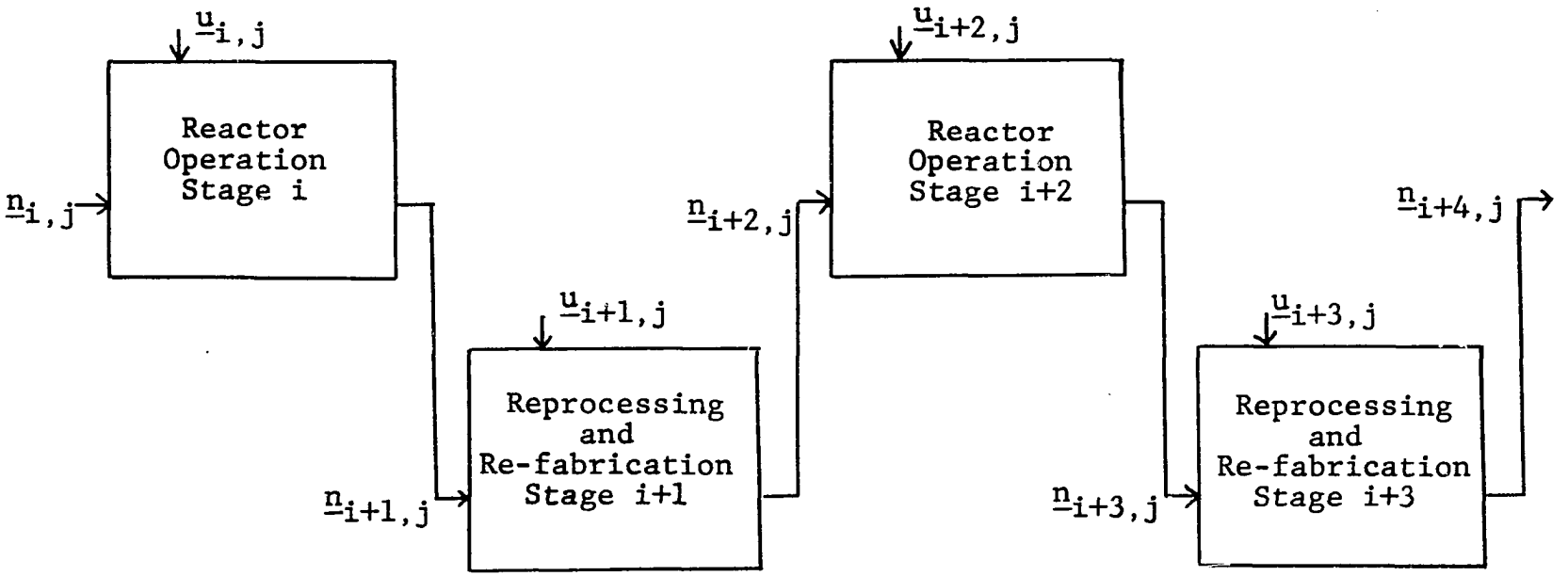

Figure 1. The multi-stage decision process flow diagram

densities for the uranium and plutonium isotopes considered. The p-dimensional decision vector which determines the operating conditions and the geometry of the fuel is denoted as $\underline{u}_{i,j}$. The final stage is denoted as i=N.

The state vector of the fuel is defined mathematically as

$$\underline{n}_{i+1,j} = \underline{g}_{i,j}(\psi_i , \pi_i) , \quad i= 1, 2, \ldots, N, \quad j=1, 2, \ldots, J \tag{1a}$$

$$\underline{n}_{1,j} = \underline{n}^{o}_{j} , \quad j=1, 2, \ldots, J \tag{1b}$$

where

$\psi_i = \{\underline{n}_{i,j} \mid j=1, 2, \ldots, J\}$

= the set of state vectors for all fuel batches

$\pi_i = \{\underline{u}_{i,j} \mid j=1, 2, \ldots, J\}$

= the set of decision vectors for all fuel batches

If Π_i is defined as the set of π_k for stages i to N, i.e.,

$$\Pi_i = \{\pi_k \mid k=i, i+1, \ldots, N\}$$

and Ψ_i is defined as the set of ψ_k for stages i to N, i.e.,

$$\Psi_i = \{\psi_k \mid k=i, i+1, \ldots, N\}$$

the performance index for a trajectory with initial condition Ψ_I and decision schedule Π_I is given as

$$PI_I = \sum_{j=1}^{J} PI_{I,j}(\Psi_I , \Pi_I) = \sum_{j=1}^{J} \Bigg[c^{i}_{I,j}(\psi_I) + \sum_{\substack{i=I \\ i \text{ even}}}^{N-1} c^{e}_{i,j}(\psi_i , \pi_i) + \sum_{\substack{i=I \\ i \text{ odd}}}^{N-1} c^{o}_{i,j}(\psi_i , \pi_i) + c^{f}_{N,j}(\psi_N , \pi_N) \Bigg] \tag{2}$$

where

$c^{i}_{I,j}$ = the initial cost contributions prior to stage I to the total performance index of fuel batch j

$c^{e}_{i,j}$ = the cost contributions to the total performance index of fuel batch j incurred after an even stage i and prior to an odd stage i+1

$c^{o}_{i,j}$ = the cost contributions to the total performance index of fuel batch j incurred after an odd stage i and prior to an even stage i+1

$c^{f}_{N,j}$ = the final cost contributions after the fuel stage N to the total performance index of fuel batch j

Note that the performance index may be thought of as a single-stage decision process, stage I, connected to a multi-stage decision process, stages I+1 to N. The performance index may then be expressed mathematically as

$$PI_I = \sum_{j=1}^{J} \left[c^{i}_{I,j}(\psi_I) + c^{f}_{I,j}(\psi_I , \pi_I) + PI_{I+1,j}(\Psi_{I+1} , \Pi_{I+1}) \right]$$

$$= \sum_{j=1}^{J} \left[f_{I,j}(\psi_I , \pi_I) + PI_{I+1,j}(\Psi_{I+1} , \Pi_{I+1}) \right] \qquad (3)$$

where

$$f_{I,j}(\psi_I , \pi_I) = c^{i}_{I,j}(\psi_I) + c^{f}_{I,j}(\psi_I , \pi_I)$$

Upon expanding PI_I in a Taylor series one has

$$PI_I = \sum_{j=1}^{J} \left\{ PI^+_{I,j} + \left[\underline{PI}^+_{x\,I,j} \right]^T \delta \underline{n}_{I,j} + \right.$$

$$\left. 1/2 \delta \underline{n}^T_{I,j} \left[PI^+_{xx\,I,j} \right] \delta \underline{n}_{I,j} + \ldots \right\} \tag{4}$$

where the superscript "+" indicates a nominal trajectory of state vectors, Ψ^+_I, and decision vectors, Π^+_I, and

$PI^+_{I,j}$ = the performance index evaluated for the nominal trajectory

$PI^+_{x\,I,j}$ = a q-dimensional column vector whose r^{th} component is $\dfrac{\partial}{\partial [\underline{n}_{I,j}]_r} PI_{I,j} \Big|_{\underline{n}_{I,j} = \underline{n}^+_{I,j}}$

$PI^+_{xx\,I,j}$ = a qxq symmetric matrix whose rs^{th} component is

$$\frac{\partial^2}{\partial [\underline{n}_{I,j}]_r \, \partial [\underline{n}_{I,j}]_s} PI_{I,j} \Big|_{\underline{n}_{I,j} = \underline{n}^+_{I,j}}$$

One needs to calculate the Taylor series coefficients $PI^+_{I,j}$, $\underline{PI}^+_{x\,I,j}$ and $PI^+_{xx\,I,j}$ in order to search a neighborhood of the set of state vectors, ψ^+_I, representing the number densities of a possible feasible solution so that a better feasible solution can be found.

Differentiation of Equation (3) for a given fuel batch results in the iterative equation

$$\underline{PI}_{x\,I,j}(\Psi_I, \Pi_I) = \underline{f}_{x\,I,j}(\psi_I, \pi_I) + \left[G_{x\,I,j}(\psi_I, \pi_I)\right]^T \underline{PI}_{x\,I+1,j}(\Psi_{I+1}, \Pi_{I+1}) \tag{5}$$

where

$\underline{f}_{x\,I,j}$ = a q-dimensional column vector whose r^{th} component is $\dfrac{\partial}{\partial[\underline{n}_{I,j}]_r} f_{I,j}$

$G_{x\,I,j}$ = a qxq dimensional matrix whose r^{th} column is the vector $\dfrac{\partial}{\partial[\underline{n}_{I,j}]_r} g_{I,j}$

It is convenient to avoid notational difficulties by introducing a pseudo-Hamiltonian function defined by

$$h_{I,j}(\psi_I, \pi_I, \lambda) = f_{I,j}(\psi_I, \pi_I) + \lambda^T g_{I,j}(\psi_I, \pi_I) \tag{6}$$

Then Equation (5) may be written as

$$\underline{PI}_{x\,I,j}(\Psi_I, \Pi_I) = \underline{h}_{x\,I,j}(\psi_I, \pi_I, \underline{PI}_{x\,I+1,j}(\Psi_{I+1}, \Pi_{I+1})) \tag{7}$$

where

$\underline{h}_{x\,I,j}$ = a q-dimensional column matrix whose r^{th} component is $\dfrac{\partial}{\partial[\underline{n}_{I,j}]_r} h_{I,j}$

Differentiation of Equation (7) again results in

$$PI_{xx\,I,j}(\Psi_I, \Pi_I) = H_{xx\,I,j}(\psi_I, \pi_I, \underline{PI}_{x\,I+1,j}(\Psi_{I+1}, \Pi_{I+1}))$$
$$+ \left[G_{x\,I,j}(\psi_I, \pi_I)\right]^T \left[PI_{xx\,I+1,j}(\Psi_{I+1}, \Pi_{I+1})\right]\left[G_{x\,I,j}(\psi_I, \pi_I)\right] \quad (8)$$

where

$H_{xx\,I,j}$ = a qxq dimensional matrix whose rs^{th} component is

$$\frac{\partial^2}{\partial[\underline{n}_{I,j}]_r \; \partial[\underline{n}_{I,j}]_s} \; h_{I,j}$$

In summary, iterative expressions have been derived for the coefficients of the truncated Taylor series. For a given nominal state these are

$$PI^+_{I,j}(\Psi^+_I, \Pi^+_I) = f^+_{I,j}(\psi^+_I, \pi^+_I) + PI^+_{I+1,j}(\Psi^+_{I+1}, \Pi^+_{I+1}) \quad (9a)$$

$$PI^+_{x\,I,j}(\Psi^+_I, \Pi^+_I) = \underline{h}^+_{x\,I,j}(\psi^+_I, \pi^+_I, \underline{PI}^+_{x\,I+1,j}(\Psi^+_{I+1}, \Pi^+_{I+1})) \quad (9b)$$

and

$$PI^+_{xx\,I,j}(\Psi^+_I, \Pi^+_I) = H^+_{xx\,I,j}(\psi^+_I, \pi^+_I, \underline{PI}^+_{x\,I+1,j}(\Psi^+_{I+1}, \Pi^+_{I+1}))$$
$$+ \left[G^+_{x\,I,j}(\psi^+_I, \pi^+_I)\right]^T \left[PI^+_{xx\,I+1,j}(\Psi^+_{I+1}, \Pi^+_{I+1})\right]\left[G^+_{x\,I,j}(\psi^+_I, \pi^+_I)\right] \quad (9c)$$

The State Vector Function

In order to determine the coefficients given by Equations (9), it is necessary to determine $G^+_{x\,I,j}(\psi^+_I, \pi^+_I)$, the derivatives of the state vector function defined by Equation (1). The vector function $g_{i,j}(\psi_i, \pi_i)$ may be determined from a

matrix solution of the system of nonlinear differential equations describing the transmutation processes which occur both during the in-core and the out-of-core cycle time periods.

The transmutation or burnup equation for the j^{th} fuel batch and the k^{th} time step within the i^{th} stage is given by

$$\frac{d}{dt}\,\underline{n}_{i_k,j}(t) = A_{i_k,j}(\psi_{i_k}, \pi_{i_k}, t)\,\underline{n}_{i_k,j}(t) \tag{10}$$

where $A_{i_k,j}$ has been denoted as a burn matrix [28] and has the form

$$A_{i_k,j}(\psi_{i_k}, \pi_{i_k}, t) = \Lambda_{i_k,j} + \hat{A}_{i_k,j} \tag{11}$$

Radioactive decay transmutation is incorporated into the matrix $\Lambda_{i_k,j}$ and neutron reaction transmutation is incorporated into the matrix $\hat{A}_{i_k,j}$. The elements of $\hat{A}_{i_k,j}$ will depend upon the number of energy groups in the reactor model chosen and the particular transmutation processes considered. For a multi-energy group model the rs^{th} element of $\hat{A}_{i_k,j}$ is given as

$$[\hat{A}_{i_k,j}]_{rs} = \sum_{g=1}^{G} \sigma^{g}_{rs\ i_k,j}(\psi_{i_k}, \pi_{i_k}, t)\phi^{g}_{i_k,j}(t) \tag{12}$$

where g denotes the energy group, $\sigma^{g}_{rs\ i_k,j}$ is an appropriate-averaged cross section, and $\phi^{g}_{i_k,j}$ is the region-averaged flux in the given fuel batch. For a one group model resonance

capture may also be included in the matrix $\hat{A}_{i_k,j}$.

If only neutron absorption, capture, and fission are considered, and no resonance capture terms are considered, the matrix $A_{i_k,j}$ will be lower triangular with real, distinct eigenvalues. If, however, more complex neutron reactions, resonance capture terms, and alpha decay chains are considered, $A_{i_k,j}$ will have elements above the diagonal. The matrix can be written in a block lower triangular form.

The solution of Equation (11) is

$$\begin{aligned} \underline{n}_{i_{k+1},j} &= \exp(A_{i_k} \Delta t_{i_k}) \underline{n}_{i_k,j} \\ &= B_{i_k,j}(\psi_{i_k}, \pi_{i_k}, \Delta t_{i_k}) \underline{n}_{i_k,j} \end{aligned} \tag{13}$$

where the exponential matrix has been denoted as the single step transmutation matrix $B_{i_k,j}$ [28].

There have been many solutions proposed for obtaining this transmutation matrix $B_{i_k,j}$. The most obvious is the matrix exponential expansion [29]

$$e^{A\Delta t} = I + A\Delta t + \frac{A^2(\Delta t)^2}{2!} + \ldots \tag{14}$$

which has been used in depletion calculations using a scaling matrix and multiplication rules to reduce round-off errors [30].

One can use a similarity transformation M on A to diagonalize A. Therefore

$$MM^{-1}\, e^{A\Delta t}\, MM^{-1} = Me^{M^{-1}\, A\Delta t M}\, M^{-1} = Me^{D\Delta t}\, M^{-1} \tag{15}$$

where M and M^{-1} are the modal and inverse modal matrices, respectively, and D is a diagonal matrix. The exponential diagonal matrix is equivalent to a diagonal matrix whose diagonal elements are given as $e^{d_{rs}\Delta t}$, that is,

$$e^{D\Delta t} = \begin{bmatrix} e^{d_{11}\Delta t} & 0 & 0 & \cdots \\ 0 & e^{d_{22}\Delta t} & 0 & \\ 0 & 0 & e^{d_{33}\Delta t} & \\ \vdots & & & \ddots \end{bmatrix} \tag{16}$$

To obtain the modal matrix, it is necessary to determine the eigenvectors of the system [31]

$$A\underline{e}_r = \lambda_r \underline{e}_r \tag{17}$$

and the eigenvectors of the dual system [31]

$$A^T \underline{u}_s = \mu_s \underline{u}_s \tag{18}$$

This has been accomplished by the use of a block lower triangular matrix algorithm to obtain block off diagonal components of the eigenvectors from the block diagonal components [32]. The block diagonal components were determined by the power method and Newton's one-step correction method. With a deeper

understanding of the eigenvalue-eigenvector structure of the matrices, it is possible to develop from the power method more advanced iterative techniques [33, 34].

When A is lower triangular, however, a simple algorithm was developed to determine the eigenvectors rapidly and exactly from the equations

$$| A - \lambda_r I | = 0 \tag{19a}$$

$$| A^T - \mu_s I | = 0 \tag{19b}$$

Let the elements of the qxq dimensional matrices A, A^T, M and M^{-1} be denoted as $a_{r,s}$, $a^T_{r,s}$, $m_{r,s}$ and $m^{-1}_{r,s}$, respectively. Then

$$\begin{aligned} m_{r,s} &= 0 && r < s \\ m_{r,s} &= 1 && r = s \\ m_{r,s} &= \sum_{p=s}^{r-1} a_{r,p}\, m_{p,s} / (a_{s,s} - a_{r,r}) && r > s \end{aligned} \tag{20}$$

and

$$\begin{aligned} m^{-1}_{r,s} &= 0 && r < s \\ m^{-1}_{r,s} &= 1 && r = s \\ m^{-1}_{r,s} &= \sum_{p=s+1}^{r} a^T_{s,p}\, m^{-1}_{r,p} / (a^T_{r,r} - a^T_{s,s}) && r > s \end{aligned} \tag{21}$$

In order to reduce computation time when the A matrix has elements above the diagonal, a method was introduced by Duane [35]. In this method the transmutation matrix, B, is given by the matrix exponential expansion

$$B = e^{A\Delta t} = I + A\Delta t \left[e^{A\Delta t} - I \right] A^{-1}/\Delta t \tag{22}$$

with

$$[e^{A\Delta t} - I] A^{-1}/\Delta t = \sum_{p=0}^{\infty} A^p/(p+1)! \tag{23}$$

It is necessary to scale A by 2^N such that the sum of the squares of the diagonal elements is less than or equal to one. The necessary recursion expressions for scaling back are provided as

$$e^Z = Z[e^Z - I]Z^{-1} + I \tag{24}$$

$$1/2[e^{2Z} - I] Z^{-1} = 1/2[e^Z + I] [e^Z - I] Z^{-1} \tag{25}$$

where

$$Z = \frac{A}{2^N} \Delta t$$

Having obtained the transmutation matrix for the k^{th} time step of the i^{th} stage, the state vector at the beginning of the $i+1^{th}$ stage is given as

$$\underline{n}_{i+1,j} = \left[\prod_{k=1}^{K} B_{i_k,j}(\psi_{i_k}, \pi_{i_k}, \Delta t_{i_k})\right]\underline{n}_{i,j}$$

$$= \Phi_{i,j}(\psi_i, \pi_i, \Delta t_i)\underline{n}_{i,j} \tag{26}$$

where $\Phi_{i,j}$ is the state transition matrix [36] for the system outlined in Figure 1. In general the elements of $\Phi_{i,j}$ are autonomous, that is, they do not depend on the time explicitly. The cross section input will depend on the core composition and control and operation mode. For minor changes in the core composition and a constant control and operation mode, the state transition matrix is approximately constant. The matrix $G^{+}_{x\,I,j}$ is then estimated as

$$G^{+}_{x\,I,j}(\psi^{+}_I, \pi^{+}_I) = \Phi^{+}_{I,j}(\psi^{+}_I, \pi^{+}_I) \tag{27}$$

The Performance Index

The performance index used in this study is defined as the sum of the individual cost components of the nuclear fuel cycle from the levelized fuel cycle cost equation.

The nuclear fuel cycle

The basic components of the nuclear fuel cycle are shown in the diagram in Figure 2. Numerical values for the cost

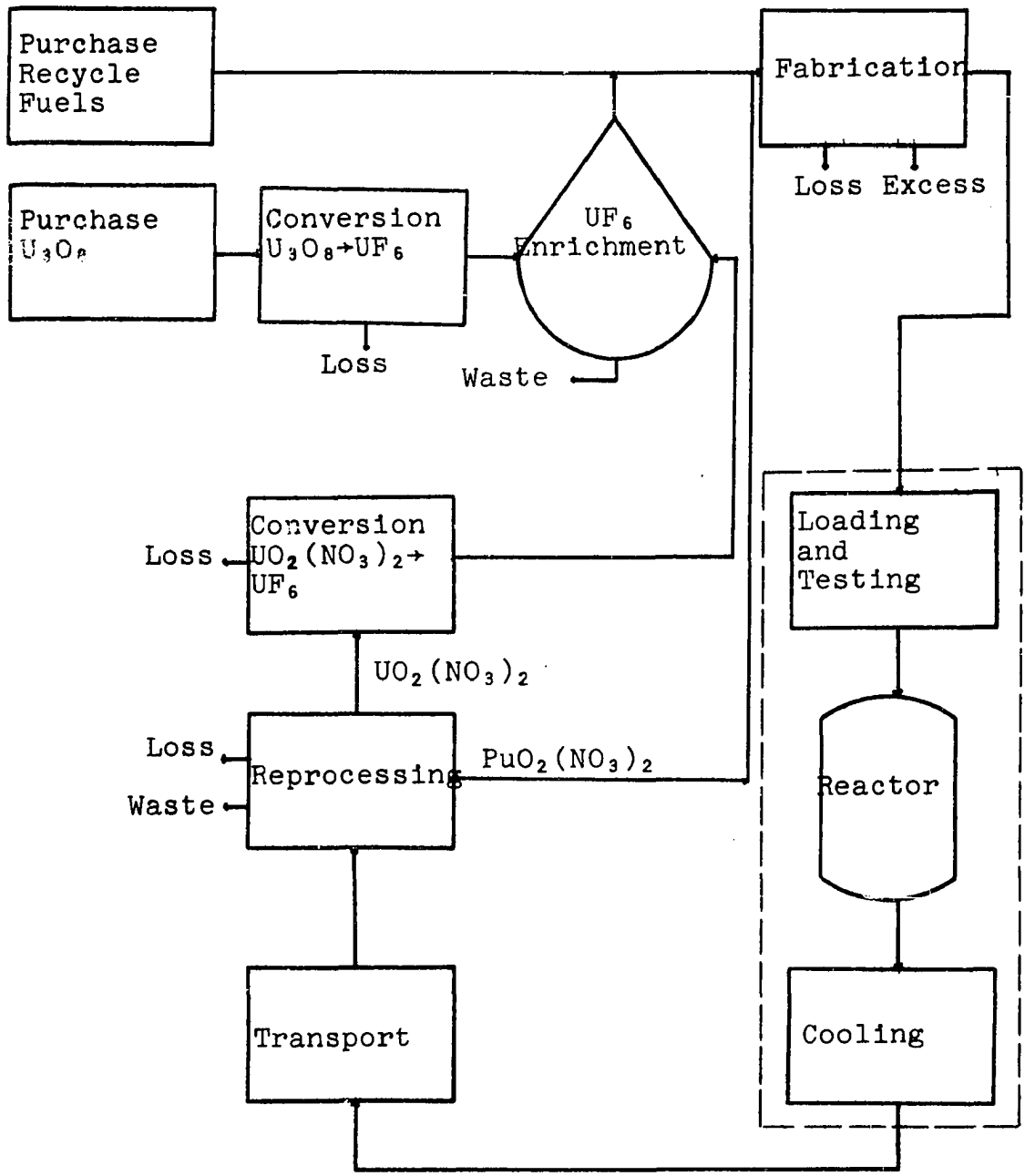

Figure 2. The nuclear fuel cycle

components were chosen to be representative of 1977 costs. Inflationary effects may be taken into account by adjustment of the rate of return parameter. For the purposes of this work interest charges and inflationary effects will be incorporated into an assumed rate parameter.

Purchase of U_3O_8 The mass of U_3O_8 which must be purchased is dependent on the feed supply required for the enrichment process, the magnitude of material losses in the conversion and fabrication processes, and the extent of fuel recycle. The market price of U_3O_8 depends on the supply and demand. In the early '70's the supply exceeded the demand, and the market price was a little less than $7 per pound. Future projections indicate this price will increase moderately [37]. A representative cost for 1977 was taken to be $12 per pound [38].

Conversion of U_3O_8 to UF_6 The enrichment of uranium-235 by the gaseous diffusion process necessitates that the U_3O_8 be converted to UF_6. For purposes of this study the cost of conversion will also include sampling costs of the ore concentrate and transportation costs. During the chemical processes of converting the ore to UF_6 there are some losses of material. The cost of these losses are included as extra mass of U_3O_8 which must be purchased. The cost of conversion is taken to be $1.25 per pound uranium [37].

Enrichment of uranium-235 The enrichment of uranium-235 is presently available only from the AEC through the

gaseous diffusion plants. A derivation for the total flow rate in an ideal cascade gaseous diffusion plant is given in Reference [39]. It is found to be the product of a factor related to separation, and a factor which is proportional to the throughput denoted as the separative duty

$$S = W\phi(x_w) + P\phi(x_p) - F\phi(x_f) \tag{28}$$

where

F, P, W = mass of feed, product and waste material, respectively

x_f, x_p, x_w = atom fraction of uranium-235 in the feed, product, and waste material, respectively

The function $\phi(x)$ is the separation potential given by

$$\phi(x) = (2x - 1)\,\ell n[x/(1 - x)] \tag{29}$$

In a gaseous diffusion plant built as an ideal cascade of stages, the total flow rate, the total pump capacity, the total power demand, and the total barrier area are all proportional to the separative duty. Therefore enrichment cost is also based upon this factor.

The total charge of enriched material is then

$$C_E = FC_F + SC_S$$

where

F = mass of feed material

C_F = cost of the feed material per mass

S = separative work (duty)

C_S = cost of enrichment per separative work unit

The diffusion plant currently operate at a tails assay of 0.003, but the USAEC charges customers for enriching services as if the plants operated at a 0.002 tails assay [40]. The difference in the amount of feed required is drawn from the USAEC stockpile of 50,000 tons of natural uranium. Therefore the AEC can dispose of surplus uranium without flooding the uranium market.

Enrichment services for uranium-235 could be contracted on a requirements basis for $38.50 per separative work unit or on a fixed basis for $36 in 1973. Starting on January 1, 1974, these charges were increased 1%; rounded upward to the nearest $0.05. These increases will be continued every January 1 and July 1 [41]. The USAEC has constructed a standard table of enriching services which includes uranium with an enrichment less than that of natural uranium [41]. The base charge for depleted uranium was $2.50 per kilogram in 1974.

Purchase of recycle fuels Recycle fuels in this study refer to plutonium and uranium isotopes. The market price of plutonium will depend upon its supply and demand as a research and development material, weapons material, light water reactor fuel, and perhaps as a fast breeder reactor fuel or makeup fuel in the high temperature gas cooled reactor. In one

projection it is stated that plutonium recycle values will likely range from $3 to $7 per gram of fissile content until 1976 and then increase to a range of $7 to $9 per gram in the early 1980's [42]. This will depend on the length of time before the fast breeder is commercially accepted. In this study a value of $6 per gram fissile was chosen as being representative of the cost for 1977 [38].

The purchase price of recycle uranium will be dependent on the cost of producing uranium of an equal enrichment less a penalty for the presence of uranium-236 which is a reactor poison. Uranium-236 also complicates the enrichment process. However, its presence in a reactor can lead to the production of plutonium-238, and the cost penalty will depend on the market price of this isotope [43].

Fabrication of the fuel elements Fabrication costs include the costs of hardware; pelletizing, shaping, and machining of the fuel material; cladding material; assembly of the fuel elements; and quality control costs. Fabrication costs for uranium-plutonium mixed oxide fuels are expected to be greater than uranium oxide fuel fabrication costs. This is because special precautions must be taken during plutonium fabrication and handling due to its toxic and radioactive characteristics. Smaller diameter fuel rods will be used in fast and thermal reactors utilizing plutonium oxide as the fuel. Smaller rods will increase fabrication costs.

It is expected that the fabrication of mixed oxide fuels will be 20 to 100 percent higher than uranium oxide fuels. The fabrication costs are assumed to be $70 per kgU for uranium oxide rods in thermal reactors and $500 per kg heavy component for fast reactor core rods and $200 per kg heavy component for blanket rods [37].

Credit for excess material The excess material from fabrication is credited to the customer. The amount of this credit will depend upon the isotopic composition of the fuel.

Operation and cooling During the operation of the reactor, the fuel is depreciated as the fissile material is used to generate electricity. There are a number of methods which may be used to calculate the depreciation of the fuel. The method used in this study is the sum-of-the-years depreciation [44].

Transport of spent fuel Transportation of the spent fuel requires special casks. The casks are designed to contain, shield, and cool the radioactive material. Transportation costs are about $5 per kg heavy component discharged [37].

Reprocessing of the spent fuel Reprocessing of the nuclear fuel elements includes removal of the cladding material and separation of the fission products from the uranium and plutonium. The estimated cost for reprocessing is about $35 per kg heavy component and $200 per kg heavy component discharged for thermal and fast reactor assemblies, respectively

[37].

Credit for discharged uranium and plutonium The credit for discharged uranium is taken to be the value of fresh uranium of an equal enrichment, less a penalty for the presence of uranium-236. The poison cost penalty of uranium-236 has been estimated to be $1 per gram of uranium-236 [43].

The present worth technique and levelized fuel cycle cost equation

The entire nuclear fuel cycle requires expenditures and receipts over a period of approximately five years. It is important to consider, therefore, the alteration of the value of money as it is exchanged between the utility, customers, and creditors during the fuel cycle time period. The present worth of money is always greater than an identical amount in the future, because money may earn interest while invested during the interim. A common method used to take into account the changing worth of money is the present worth technique. In this technique the different expenditures and receipts are referred to one point in time by appropriate adjustment to reflect the potential effective earning power during the interim periods.

By using the present worth technique and by requiring that the indebtedness for fuel cycle investments be reduced to zero at the end of the fuel cycle time period, the levelized fuel cycle cost may be obtained. The levelized fuel cycle

cost is an equivalent constant charge during the fuel cycle time period. It accounts for the present worth of all cash flow during the fuel cycle. The levelized fuel cycle cost equation generalized for continuous discounting as used in this study [45] is

$$c = \frac{\dfrac{I_i - I_f e^{-r'n}}{(1 - \tau)} - \displaystyle\int_{t=0}^{n} e^{-r't} \frac{\tau}{(1 - \tau)} F_d(t)\,dt}{\displaystyle\int_{t=0}^{n} e^{-r't} Q(t)\,dt} \qquad (30)$$

where

I_i = sum of the present worth of all the depreciable investments referenced to the start of reactor operation (mills/kg)

I_f = present worth of the salvage value referenced to the end of reactor operation (mills/kg)

$F_d(t)$ = depreciation during the t^{th} month (mills/month-kg)

$Q(t)$ = quantity of electricity generated in t^{th} month (Kwh(e)/month-kg)

τ = income tax rate

r' = effective rate of return

n = length of the reactor operating time period for the given fuel loading (months)

To account for interest and inflation changes, the effective rate of return may be adjusted. An increase in the interest rates will increase the effective rate of return, while an increase in the inflation rate will decrease this parameter [46].

PROCEDURE

The systematic procedure for solving a nuclear fuel management in-core optimization problem using DDP is presented schematically in Figure 3. Details of each step are presented below.

The procedure begins by defining the problem and a nominal trajectory. The beginning nominal trajectory, for example, may be one which has been used in the past or some hypothetical trajectory. The nominal trajectory must be feasible, however.

Having chosen a nominal trajectory, the physics calculations are completed, from which the performance index and the Taylor series coefficients are obtained. In the depletion calculations the matrix exponential technique as discussed earlier was used. These techniques will accommodate any degree of complexity in the physics calculations. The physics calculations in this study were computed using a modified one energy group or a four energy group multi-zone neutron balance model. The power distributions using these models were verified by diffusion calculations. Cross sections were determined from a variety of procedures including averaging over a Maxwellian, Wigner-Wilkins, and Nelkin flux spectrum. The specific application of these procedures are discussed later.

From the physics calculations the performance index and the Taylor series coefficients are determined. A search is then conducted about the nominal trajectory to determine the

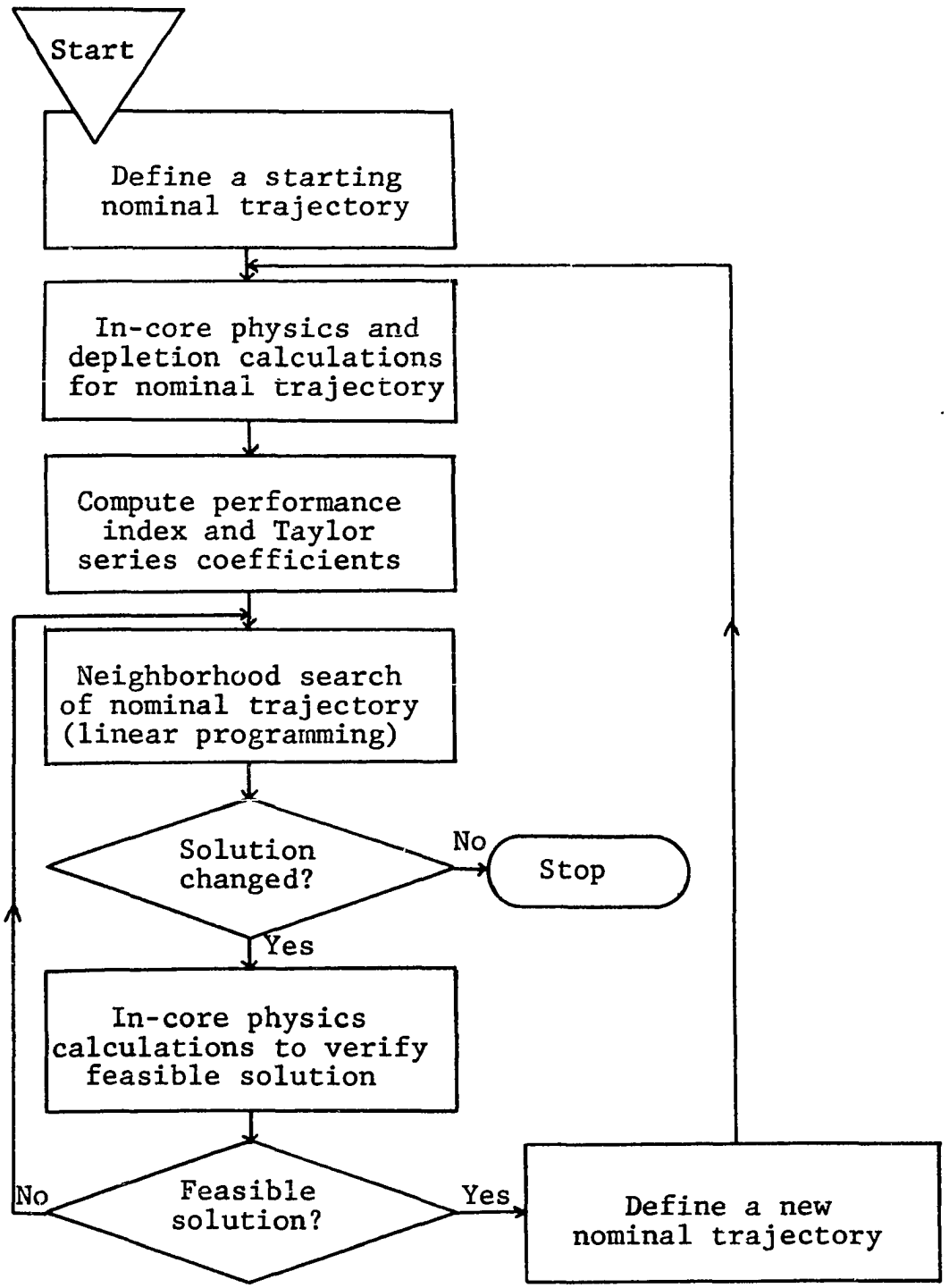

Figure 3. Systematic procedure for solving a fuel management optimization problem using differential dynamic programming

optimal trajectory within the bounds of the search domain. A variety of search methods are available [47]. In this study a uniform search was chosen. The third term of the Taylor series expansion given by Equation (4) yields only a small contribution to the total. This effect will be discussed later. The optimization problem about the neighborhood of the nominal trajectory is then a linear programming problem. The large number of constraints, however, makes this problem difficult to solve using a formal method such as Simplex [48]. These constraints include criticality, conservation of mass, blending, reactivity, and convergence constraints in the forms of inequalities, equalities, and constant ratios. Hence, it was found easier to conduct a uniform search consistent with these constraints in determining the optimum.

If a better solution exists, it is necessary to determine if that solution is feasible with respect to the constraints and the assumptions made in the analysis. The constraints used in the search were approximated and must be confirmed by detailed calculations at this point. Other constraints imposed upon the solution but not considered during the search include maximum burnup and gross-power peaking.

Two common assumptions used in previous studies [6, 12, 13, 16] and used for certain problems considered in this study are

1. Fuel (lattice) burnup calculations can be separated

from the spatial burnup problem.

2. The power control problem may be separated from the economic optimization problem.

The first assumption is often used to reduce the complexity of the problem due to a lack of a procedure to handle zone burnup changes as a function of the isotopic composition and control capabilities. The independent variables (ψ_I , π_I) used in Equations (1) to (27) may be replaced by ($\underline{n}_{i,j}$, $\pi_{I,j}$) where $\pi_{I,j}$ contains only decision vectors in the region of the j^{th} fuel batch. The second assumption is reasonable as it has been noted [6, 13] that fuel management decisions generally have a greater effect on burnup maximization than poison management decisions.

The calculations used to determine the feasibility of a solution may or may not be more complex than those calculations used to complete the depletion calculations, depending on the complexity of the constraints. Having determined an optimal feasible solution, it is used as the new nominal trajectory, and the process is repeated until no better solution can be found.

As previously mentioned, the solution may not be a global optimum. This depends on the shape of the performance index surface when determined as a function of the independent variables. It also depends on the functional relationship used to determine the performance index. There is disagreement

concerning the shape of this surface. Stoll and Axford [23], in a similar study, reported that numerical experience, although certainly not infallible, indicated only a unique optimum. Fenech [49], however, reported in a general discussion on the optimization of nuclear power plant design that the objective function is not a smooth convex surface, but has several minima and maxima. To assure a global optimum, many different starting nominal trajectories should be chosen. Unfortunately, this may reduce the efficiency of the DDP method, and hence its computational competitiveness when compared to alternate methods.

The performance index has been defined as the levelized fuel cycle costs as given by Equation (20). The cost parameters given in the previous discussion on the fuel cycle are summarized in Table 1.

Table 1. Fuel cycle cost input parameters

Item	Rate	Begin payment (months)	End payment (months)
	PRESSURIZED WATER REACTOR		
Purchase of fuel	\$ 10/lb U_3O_8 \$ 6/gm Pu fissile	0	0
Conversion[a]	\$ 1.25/lb U_3O_8	0	3
Enrichment	\$ 32/swu	6	6
Fabrication[a]	\$ 70/kgU	3	13
Credit for excess		13	13
Depreciation of fuel		14	49.5
Cooling of fuel		49.5	55.5
Transport of fuel	\$ 5/kgH	55.5	55.5
Reprocessing[a]	\$ 35/kgH	55.5	57.5
Credit for excess		57.5	57.5
Fabrication penalty	.30		
	LIQUID METAL FAST BREEDER REACTOR		
Purchase of fuel	\$ 6/gm Pu fissile	0	0
Fabrication			
Core	\$500/kgH	0	10
Radial blanket	\$200/kgH	0	10
Credit for excess		10	10
Depreciation of fuel		11	22.5
Cooling of fuel		22.5	28.5
Transport of fuel	\$ 5/kgH	28.5	28.5
Reprocessing			
Core	\$200/kgH	28.5	30.5
Radial blanket	\$100/kgh	28.5	30.5
Credit for excess		30.5	30.5
	BOTH REACTORS		
Income tax rate	.5		
Effective rate of return	.07[b]		

[a] Assumed linear payments.

[b] Reference [50].

RESULTS

Comparison of the Terms in the Taylor Series Expansion

The second and third terms of the truncated Taylor series expansion given by Equation (4) were compared for a typical pressurized water reactor (PWR) batch loaded with fresh 3% enriched uranium dioxide fuel. The relative magnitude of these terms will depend on the size of the change allowed in $\delta\underline{n}$, that is, the bounds or domain of the neighborhood during the search for the new nominal trajectory. The larger the elements of $\delta\underline{n}$, the smaller the ratio of the second to third terms. An exaggerated change was made in $\delta\underline{n}$ by increasing the enrichment of the reactor to 4% uranium-235. The reactor physics model was a four group neutron balance equation with cross sections computed by LEOPARD [51, 52], which averages the cross sections over a Wigner-Wilkins spectrum [53]. The pressurized water reactor design was taken from Reference [44]. The results are presented in Table 2. The third term is always less than one percent of the second term for each cost component considered. On the basis of these results, it was decided to ignore the third term, as this would reduce and simplify computations without an appreciable effect on the accuracy.

To test the accuracy with only the first and second term of the expansion retained, the same reactor design and calculational models were used to determine the change in the

Table 2. Comparison of the relative magnitude of the Taylor series coefficients in Equation (4) for a change in enrichment from 3 to 4% uranium-235 in a batch loaded PWR

Cost component	Ratio of second to third term in Equation (4)
Purchase	108:1
Conversion	108:1
Enrichment	114:1
Fabrication	N/A[a]
Credit for Excess	111:1
Transport	N/A[a]
Reprocessing	N/A[a]
Credit for uranium	176:1
Credit for plutonium	219:1

[a] Not applicable since the first and second derivatives are zero.

levelized fuel cycle costs with enrichment changes for batch loading. The length of the reactor operating cycle was fixed at three years. Starting with an enrichment of 2.5% uranium-235, the levelized fuel cycle costs were computed as 1.86 mills/Kwh. Using this point as a nominal point, the DDP prediction of the levelized fuel cycle costs for other enrichments was determined to be the solid line shown in Figure 4. To check this prediction the actual levelized fuel cycle costs were computed for enrichments of 2.3, 2.4, 2.45, 2.55, 2.6 and 2.7% uranium-235, and are shown as circles on Figure 4. The

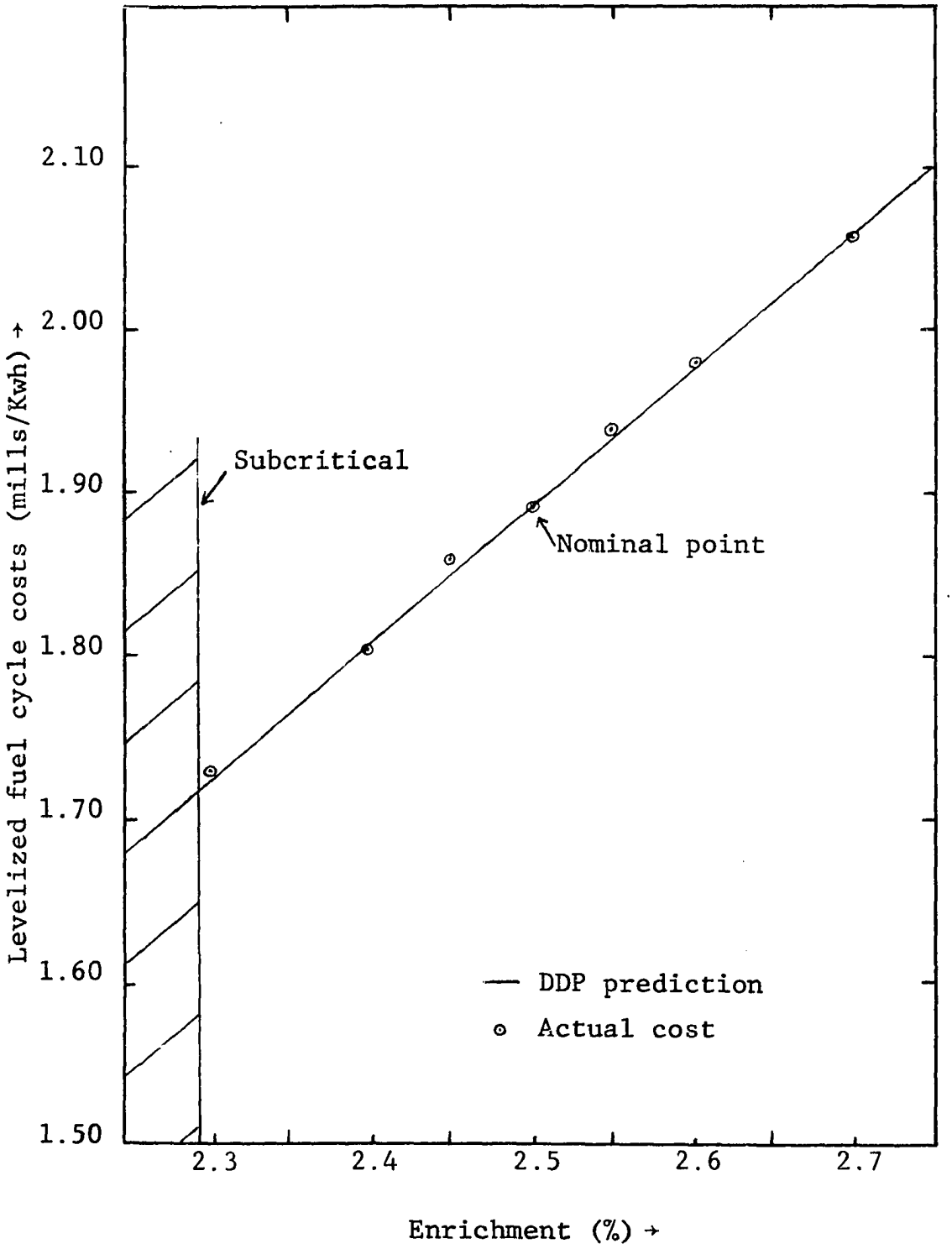

Figure 4. Levelized fuel cycle costs for a batch loaded PWR with fresh uranium dioxide

agreement between the DDP predicted costs and the actual costs is very good. One iteration was sufficient to calculate these costs over an enrichment range from 2.3 to 2.7% uranium-235. Below 2.3% uranium-235 the reactor became subcritical before the three year reactor operating cycle was over.

While the results of this study are intuitively obvious, the real purpose was to illustrate and check the method for a simple example that is easily understood. The example could have practical value for predicting how the levelized fuel cycle costs would change for enrichment changes, assuming this fuel loading was one region out of many, and assuming the power density could be specified.

When a reactor has many regions and is partially reloaded at each refueling stage, the DDP solution becomes difficult to determine if the assumption that the fuel region burnup problem may be separated from the core burnup problem is not used. Consider a reactor with three radial fuel regions and an out-in fuel management policy with the center region discharged yearly. The same design and physics models were used in conducting the analysis for this reactor. Uniform control was assumed, but the above assumption concerning the separation of the fuel region burnup problem from the core burnup problem was not assumed. The reactor was operated for one year with a center region enrichment of 2.5% uranium-235, and increasing enrichments of 2.75 and 3% uranium-235 for the other regions.

At the end of the year the center fuel region was removed and reprocessed. The levelized fuel cycle costs for this region were found to be 4.21 mills/Kwh. If burnup changes are not considered, a change in enrichment will give the DDP prediction as given by the solid line in Figure 5. A few actual points were calculated for comparison as given by the circles. The agreement is poor and was assumed to be due to burnup changes, which have not as yet been considered.

As the enrichment of the center region is changed, the relative fission rates between the three regions also changes when uniform control is assumed. Hence, the total burnup of each region changes. No method of accounting for these burnup changes was found in the literature. However, an attempt to account for these changes was formulated by noting that the levelized fuel cycle costs consist of a ratio of money to energy produced. The energy produced is directly proportional to the burnup. The relative burnup compared to other regions is approximately proportional to the ratio of the product of the flux and fissile number densities in the region being considered to the total volume average of this product. Hence, if

$$C = \text{costs} = \frac{M}{E} = \frac{\text{money}}{\text{energy produced}} \tag{31}$$

then

$$C' = \frac{dC}{dn} = \frac{M'}{E} - \frac{M}{E^2}E' = \frac{M'}{E} - C\left(\frac{E'}{E}\right) \tag{32}$$

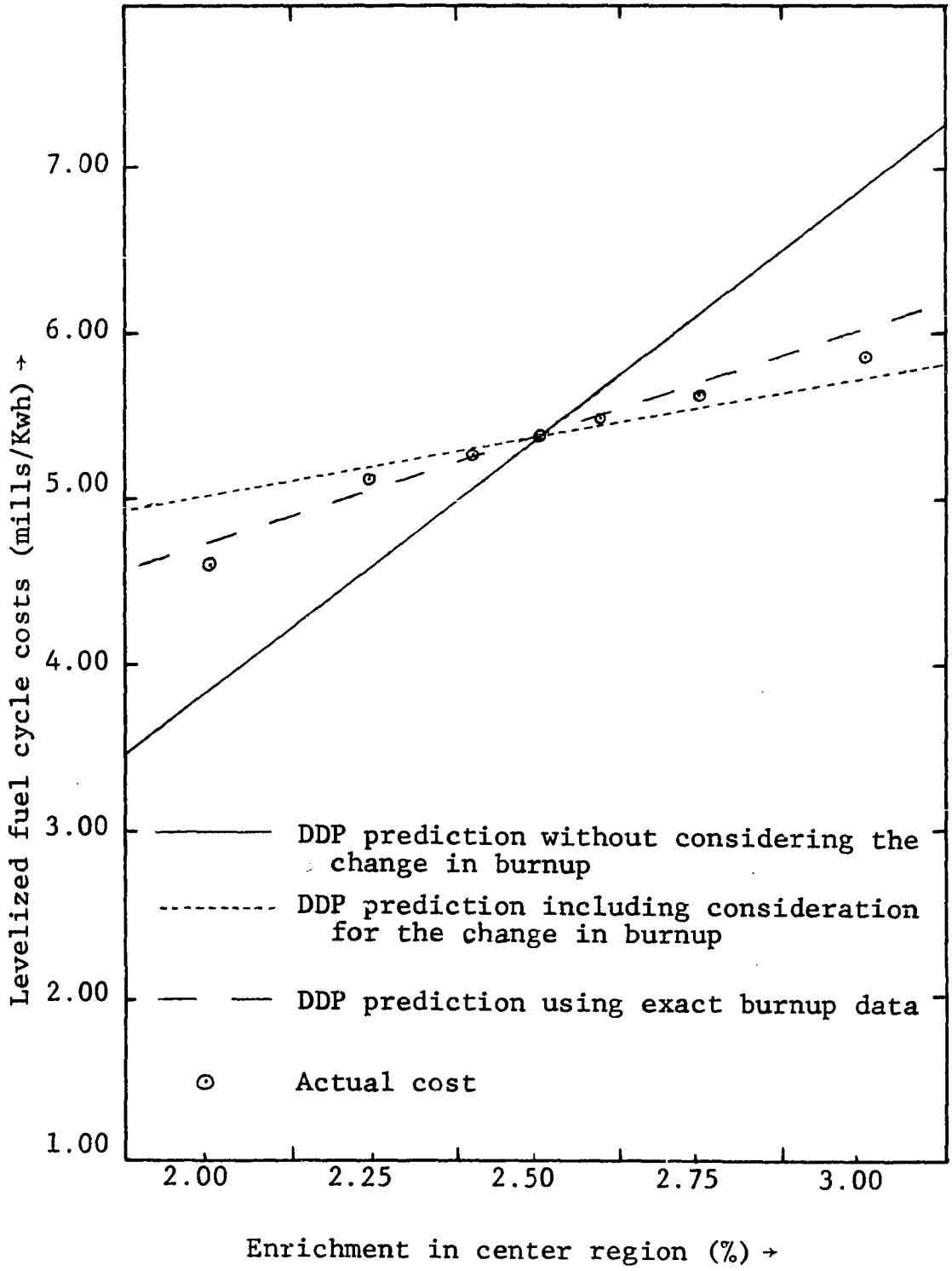

Figure 5. Levelized fuel cycle costs as a function of enrichment for the center region

where primes indicate derivatives of the state vector. Previously, the second term had been ignored, since the burnup was constant and E′ was zero. Now

$$E = \text{constant} \times \text{burnup} \simeq E_0 \frac{\phi_j n_{25_j}}{\frac{1}{J}\sum_{j=1}^{J} \phi_j n_{25_j}} \tag{33}$$

and

$$E' = \frac{dE}{dn_{25}} = E\left\{\frac{1}{n_{25_j}} - \frac{\phi_j}{\sum_{j=1}^{J} \phi_j n_{25_j}}\right\} \tag{34}$$

where j indicates the region, n_{25_j} is the initial volume averaged number density of uranium-235, and ϕ_j is the initial flux in the fuel. Using Equation (34) an estimate for the effect of burnup on the levelized fuel cycle cost was obtained, given by the dotted line as indicated in Figure 5.

The agreement between the DDP prediction and the actual costs is better, but still could be improved. The enrichment change could be changed by about 0.1% uranium-235 before serious error occurs. It was expected that the disagreement was a result of errors in the estimate of the second term in Equation (32). To test this hypothesis, the actual burnup data were plotted in Figure 6. The burnup data are nearly given by a straight line, the slope being the derivative. Calculating the burnup effect using these data confirmed the hypothesis.

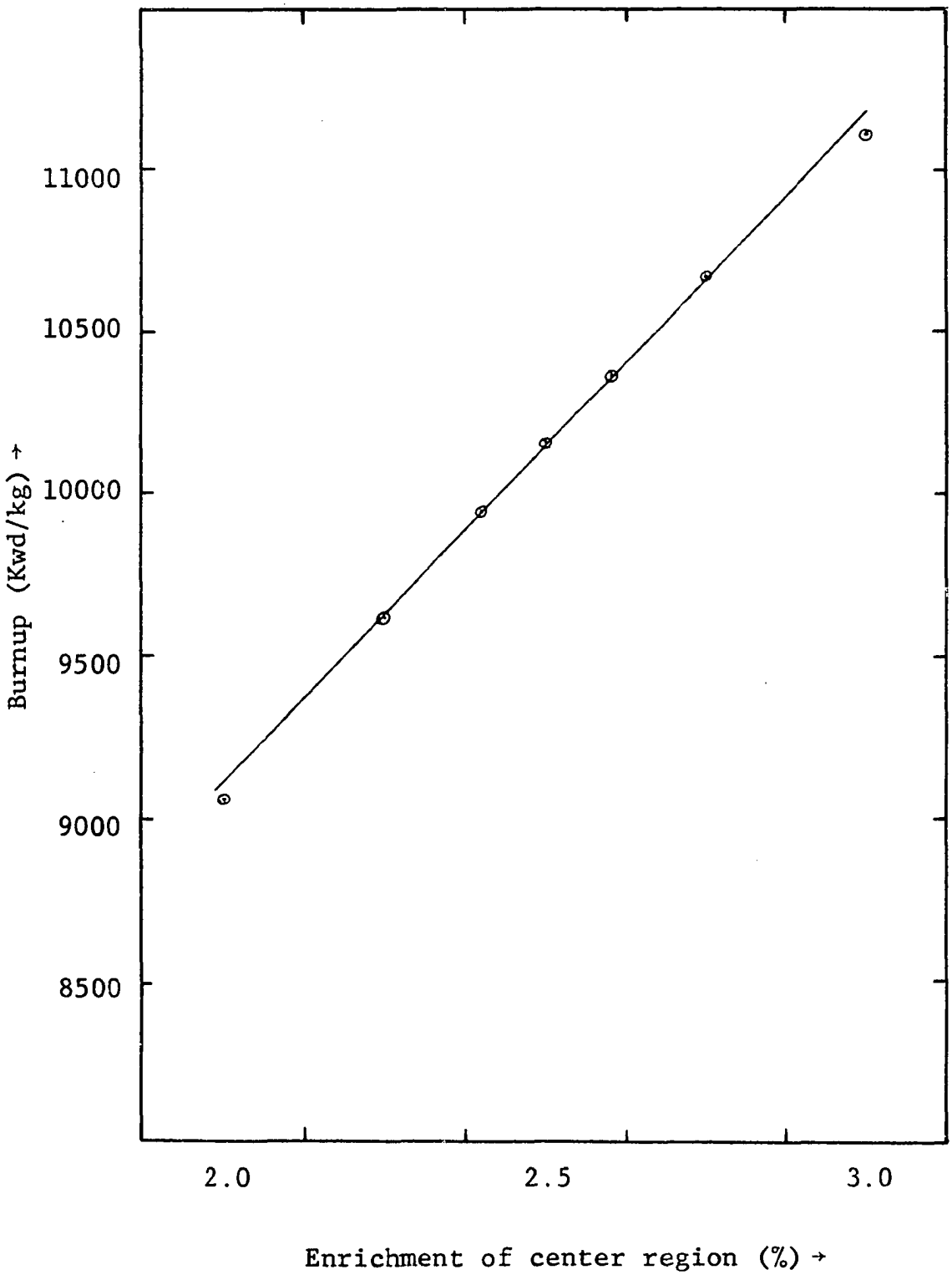

Figure 6. Burnup in center region as a function of enrichment

The results are shown as the dashed line in Figure 5.

While Equation (34) is useful in indicating how burnup changes may effect the fuel cycle costs in a given region, when an attempt to determine how other regions are effected, the problem becomes complicated. Consider, as a simple example, a two region reactor which is reloaded after one year and which has all its fuel discharged after two years. The result of a change of enrichment in fuel batch 1, as illustrated in Figure 7, will cause changes in the cost of fuel cycle components for fuel batch 1, and burnup changes in all fuel batches. The cost component changes can be accurately determined, and the burnup change can be estimated for fuel batch 1. The burnup change for the first year may be estimated from Equation (34) for fuel batch 2. It is difficult, however, to estimate burnup changes for fuel batches 2 and 3 during the second year. If the burnup of 2 is greater during the first year than in the previous examples, it will not necessarily remain greater during the second year. Therefore, fuel batch 3 being fresh fuel would be expected to achieve a higher burnup. The effect in the buildup of plutonium isotopes also complicates the analysis. Nevertheless, useful results were obtained for this example by considering how the costs would change for those cost and burnup changes that could be estimated. For enrichments of 3% uranium-235 for all fuel batches, the levelized fuel cycle costs for all batches was 4.12 mills/Kwh. The DDP

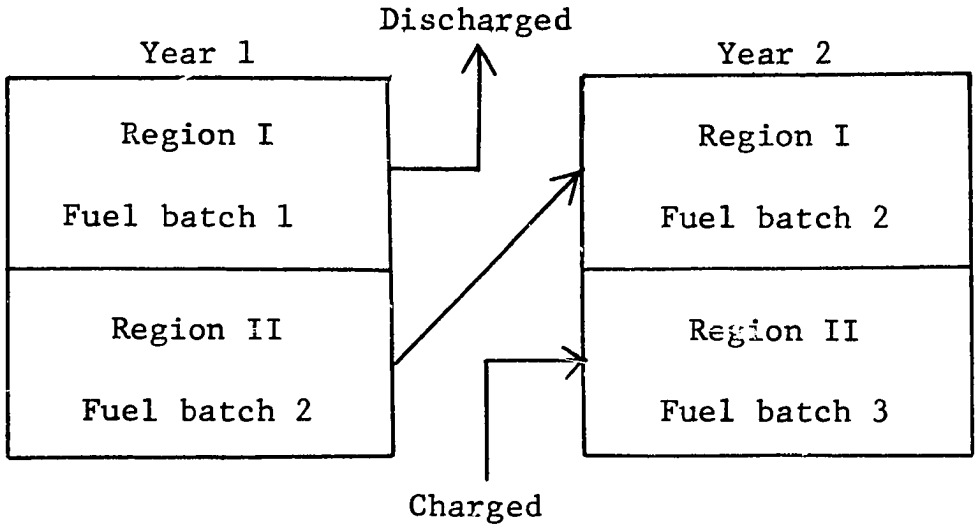

Figure 7. Two region reactor refueling scheme

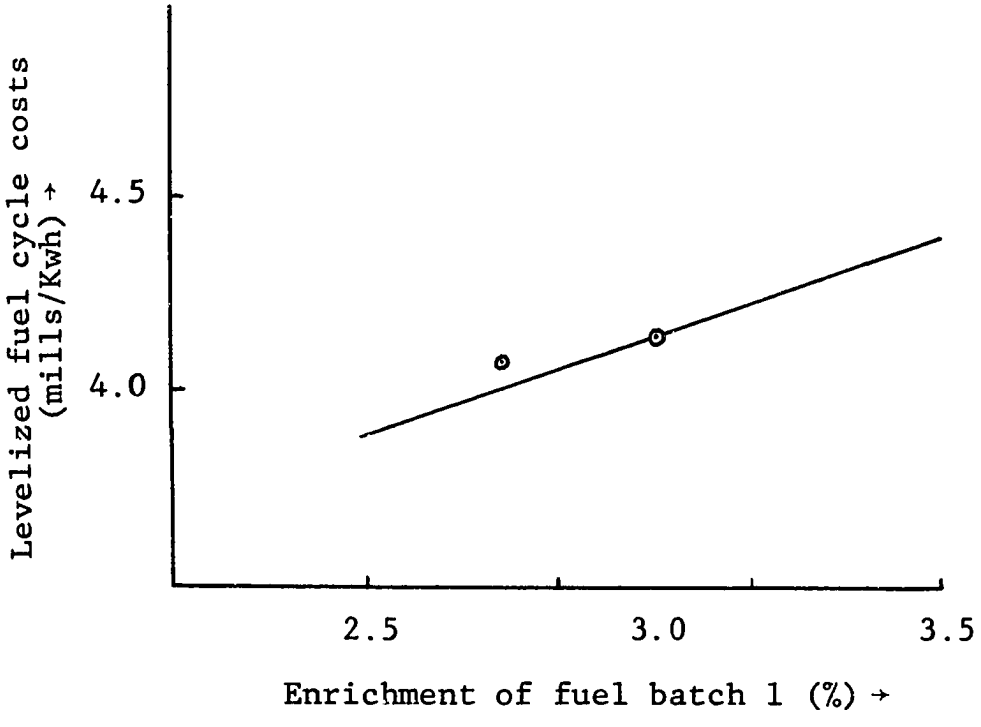

Figure 8. Levelized fuel cycle costs of fuel batch 1 as a function of enrichment

cost prediction using this point as the nominal point is given as the solid line in Figure 8. The levelized fuel cycle cost when the enrichment of fuel batch 1 was decreased to 2.75% was then computed and compared to the prediction as shown in Figure 8. The accuracy is sufficient for the procedure to be useful.

When many fuel regions are considered, the procedure becomes more complicated and is expected to be less accurate. When plutonium recycle is considered, n_{25_j} in Equation (34) must be replaced by another parameter, for example, the total fissile atom density for all fissionable isotopes weighted by the microscopic fission cross sections.

It should be noted that in the above discussion uniform control in the reactor was assumed. In the case where uniform control is not assumed, estimating the burnup changes becomes even more difficult. The burnup changes will then depend on the results of a control optimization program or a more arbitrary placement of the control material in the reactor.

Since other fissile isotopes do build up and the control material is not uniform in a practical reactor, it is very difficult to predict burnup changes. If the power density can be specified for each region over a given time period, then the burnup will be constant in each region. This can be accomplished by properly adjusting the control materials, provided the power densities are reasonable and the change in the core fissile distribution is not too great. Both of these condi-

tions need to be satisfied when iterating in the DDP procedure. Therefore, to obtain useful results it was assumed for the remainder of this study that the fuel (lattice) burnup problem could be separated from the core burnup problem.

Plutonium Recycle in a Thermal Reactor

The method of DDP was applied to a single fuel region in a PWR over a single fuel cycle of three years. The fuel for the region consisted of uranium and recycled plutonium. The plutonium isotopic composition was determined from a graph in Reference [54], which gives the expected composition as a function of reactor type and burnup. The PWR design was the same as previously used, but the calculations were based on modified one group theory. Physics parameters such as the resonance escape probabilities, fast fission factor, and fast and thermal nonleakage probabilities were determined by basic methods [55, 56, 57].

There has been much discussion presented on the core physics behavior of plutonium in thermal reactors. Dawson [58] has reported that the Maxwellian flux approximation is inadequate for plutonium-uranium-water lattices, and suggests a Nelkin scattering model. Data are also presented giving the resonance integral of plutonium-240 as a function of the concentration of that isotope.

In this study a Nelkin scattering model [59] was used to

determine the thermal cross sections using results from the PANTHER code [60] along with some experimental measurements [61, 62]. The resonance escape probability for plutonium-240 was calculated from the experimental values of the resonance integral. It was assumed that resonance capture in uranium-238 occurs at higher energies than that in plutonium-240.

The fuel lattice region under consideration was originally fueled with 2.85% enriched uranium dioxide. The fuel cycle cost is plotted in Figure 9, with the DDP prediction given by the solid line for the case when some of the uranium-235 is replaced by plutonium with an isotopic composition typical of discharged PWR fuel with a burnup of 30,000 Kwd/kgU. For the price of plutonium given as $6 per gram fissile in Table 1, it appeared advantageous to recycle the plutonium. To avoid error from too large of a step size, the fuel cycle costs with 1.0% plutonium concentration were determined, and a second iteration DDP prediction computed as shown in Figure 9. The result indicated that mixing plutonium and natural uranium would be economically advantageous. From this point two DDP predictions were computed for recycle fuel typically expected from a PWR or BWR. The actual costs were computed for mixing plutonium and depleted uranium and plotted in Figure 9. Should more plutonium be added, the fuel cycle costs would increase as shown by the DDP prediction. If less plutonium is added, the fuel cycle costs would decrease. However, the fissile enrich-

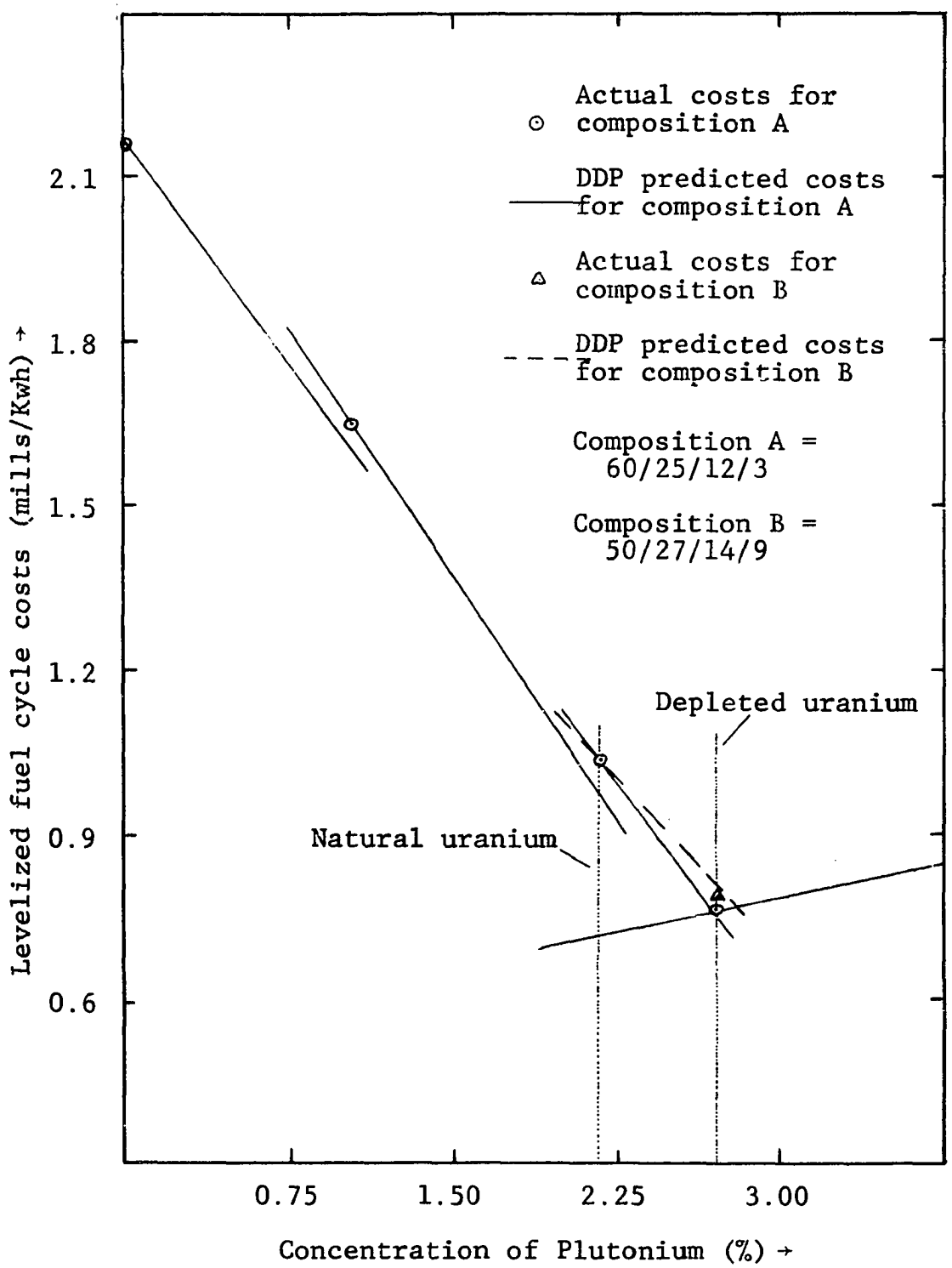

Figure 9. Levelized fuel cycle costs versus concentration of plutonium

ment in other fuel regions would have to be increased to keep the reactor critical for the entire three year cycle.

As shown in Reference [62], it is advantageous to choose depleted uranium rather than natural uranium when the price of plutonium is low. As the price of plutonium rises, it becomes desirable to use more uranium-235. The results using the cost parameters here are in agreement with those of Reference [62].

Fuel Replacement and Shuffling

Two fuel regions can be shuffled mathematically by appropriately choosing a $\delta\underline{n}$ for each region. If $\delta\underline{n}$ does not exceed the bounds for the neighborhood search, DDP may be used to predict the effects on the fuel cycle costs of a given shuffling scheme.

For this study the interchange of three fuel assemblies was considered. One assembly consisted of fresh fuel while the other two were fuel which had been irradiated from a year of reactor operation. Each assembly was subjected to slightly different operating environments and power densities which resulted in different isotopic compositions. The enrichments of each fresh fuel assembly was 3% U-235. The reactor design is the same PWR design used previously with a modified one energy group neutron balance physics model and cross sections determined from a Maxwellian flux spectrum with Wescott's non-1/v factors [56, 57].

Among the three fuel assemblies there are six possible

configurations. These are described in Table 3. Configuration 1 represents an out-in fuel management scheme and was chosen as the nominal state to conduct DDP predictions. Note that the fuel assembly in region I was replaced by a fresh fuel assembly denoted as assembly 1. Assemblies 2 and 3 represent fuel irradiated for one year in regions II and III, respectively.

Table 3. Possible fuel shuffling configurations among three fuel assemblies

	Region I	Region II	Region III
Power densities (Kw/l)	97.85	93.1	88.45
Configuration 1	Assembly 2[a]	Assembly 3	Assembly 1
Configuration 2	Assembly 1	Assembly 2	Assembly 3
Configuration 3	Assembly 1	Assembly 3	Assembly 2
Configuration 4	Assembly 3	Assembly 2	Assembly 1
Configuration 5	Assembly 2	Assembly 1	Assembly 3
Configuration 6	Assembly 3	Assembly 1	Assembly 2

[a] Assembly 1 = fresh fuel
Assembly 2 = fuel irradiated for 1 year in region II
Assembly 3 = fuel irradiated for 1 year in region III.

In Table 4 the average number densities of the isotopes in the assemblies are presented. Using these number densities and configuration 1 as the nominal state, the DDP cost predic-

Table 4. Number densities for the fuel assemblies

Assembly	Isotope	Volume averaged number densities (atoms/cm^3)
1	n_{25}	2.115×10^{22}
	n_{26}	0
	n_{28}	6.839×10^{23}
	n_{49}	0
	n_{40}	0
	n_{41}	0
	n_{42}	0
2	n_{25}	6.767×10^{21}
	n_{26}	1.262×10^{21}
	n_{28}	6.862×10^{23}
	n_{49}	1.243×10^{21}
	n_{40}	5.318×10^{20}
	n_{41}	4.066×10^{19}
	n_{42}	1.058×10^{19}
3	n_{25}	7.360×10^{21}
	n_{26}	1.173×10^{21}
	n_{28}	6.864×10^{23}
	n_{49}	1.120×10^{21}
	n_{40}	5.114×10^{20}
	n_{41}	3.748×10^{19}
	n_{42}	1.052×10^{19}

tions were determined for all the other possible configurations. The actual costs were then computed for comparison purposes. The results are given in Table 5.

From Table 5 it may be noted that the results are fairly accurate when assemblies 2 and 3 are exchanged (configuration 4), since the change in $\delta \underline{n}$ is not too great. Exchanging assembly 1 with one of the others, however, resulted in an inaccurate prediction. This is because of an extreme change in δn_{25}, as seen in Table 4. Hence fuel shuffling predictions are restricted to fuel assemblies with number densities which are not too different. From the above and preceding results, the maximum change in an element of $\underline{n}$ must be restricted to 5×10^{19} a/cm^3 averaged over the region volume. This value was used for the remainder of this study.

Plutonium Recycle Between an LMFBR and a PWR

For this study the previous PWR design and calculational models were employed. A general discussion on fuel management of liquid metal fast breeder reactors is given by Beeley et al., [63]. The reactor design was chosen as the Atomics International design [64]. A one group neutron balance model was used with cross sections from Reference [57]. The reactor was divided into three core regions, three axial blanket regions, six inner radial blanket regions and nine outer radial blanket regions. A breeding ratio of 1.18 was determined using this

Table 5. Comparison of DDP predicted cost and actual cost of various shuffling schemes

Configuration	Assembly	DDP Prediction (mills/Kwh)	Actual cost (mills/Kwh)
2	1	3.95	3.35
	2	1.58	1.58
	3	1.72	1.77
	Totals	2.45	2.20
3	1	3.95	3.35
	2	1.64	1.67
	3	1.68	1.68
	Totals	2.46	2.26
4	1	3.63	3.63
	2	1.59	1.58
	3	1.58	1.60
	Totals	2.23	2.24
5	1	4.14	3.63
	2	1.50	1.50
	3	1.72	1.77
	Totals	2.44	2.30
6	1	4.14	3.49
	2	1.64	1.67
	3	1.59	1.60
	Totals	2.44	2.25

design.

The two reactors were assumed to start up at the same time. A program for recycling plutonium between the two reactors was then formulated for a ten year time period. An out-in fuel management program was employed. The optimization problem was defined as the determination of an optimum plutonium recycle program.

To optimize the problem over the entire ten years requires that optimum refueling decisions be determined for each refueling. For this study the optimum plutonium recycle program was determined for the third and fifth refuelings of the PWR and LMFBR, respectively. The possible paths of the fuel are diagrammed in Figure 10. For each reactor discharge there are four basic paths that the discharged fuel batch may follow.

The nominal recycle program is illustrated in Figure 11. This figure represents the recycle decisions for the third refueling of the PWR and the fifth refueling of the LMFBR. Make-up fuel for later stages was held constant, although this requirement could be relaxed. If k is used to denote the recycle possibilities and $\delta \underline{n}_k$ is the change in initial number densities, the linear problem which must be solved for the optimum costs during the heighborhood search takes the form

$$\underset{\{\delta \underline{n}_k\}}{\text{Min}}\ \delta PI = \sum_k \underline{a}_k^T \delta \underline{n}_k \tag{35}$$

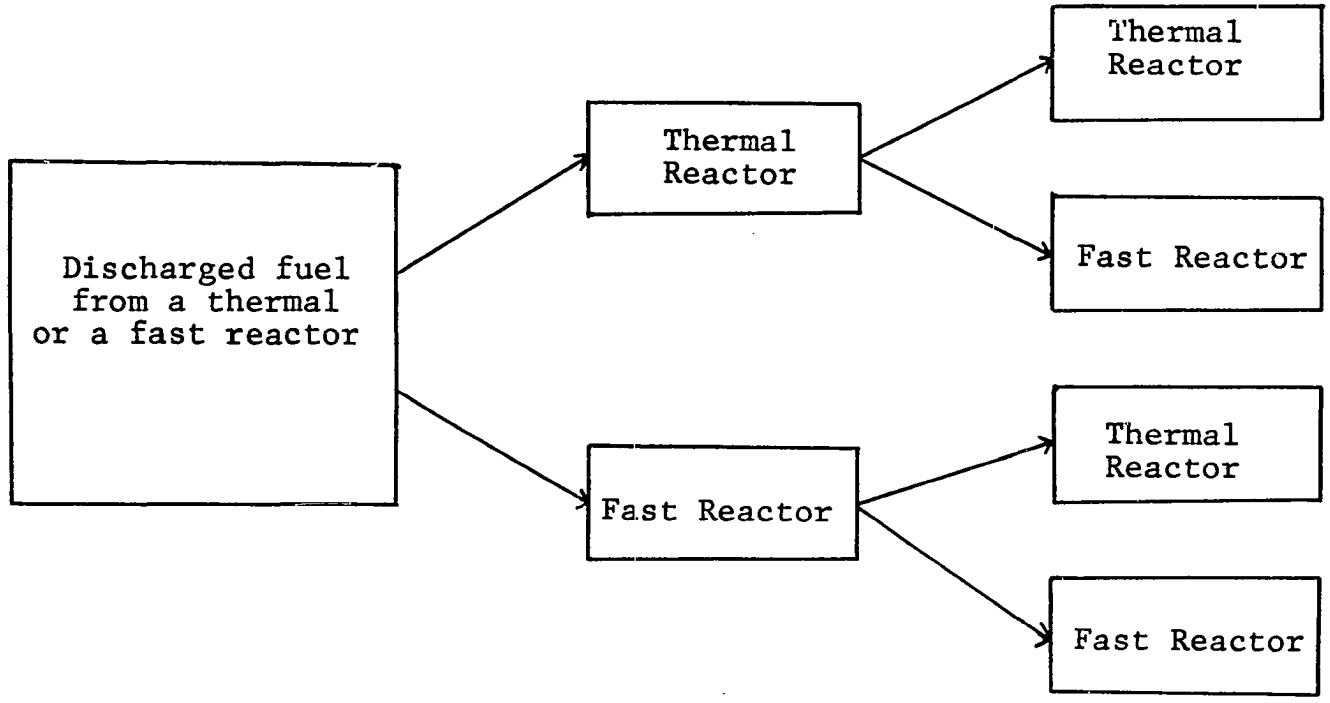

Figure 10. Recycle possibilities for discharged fuel from a thermal or a fast reactor

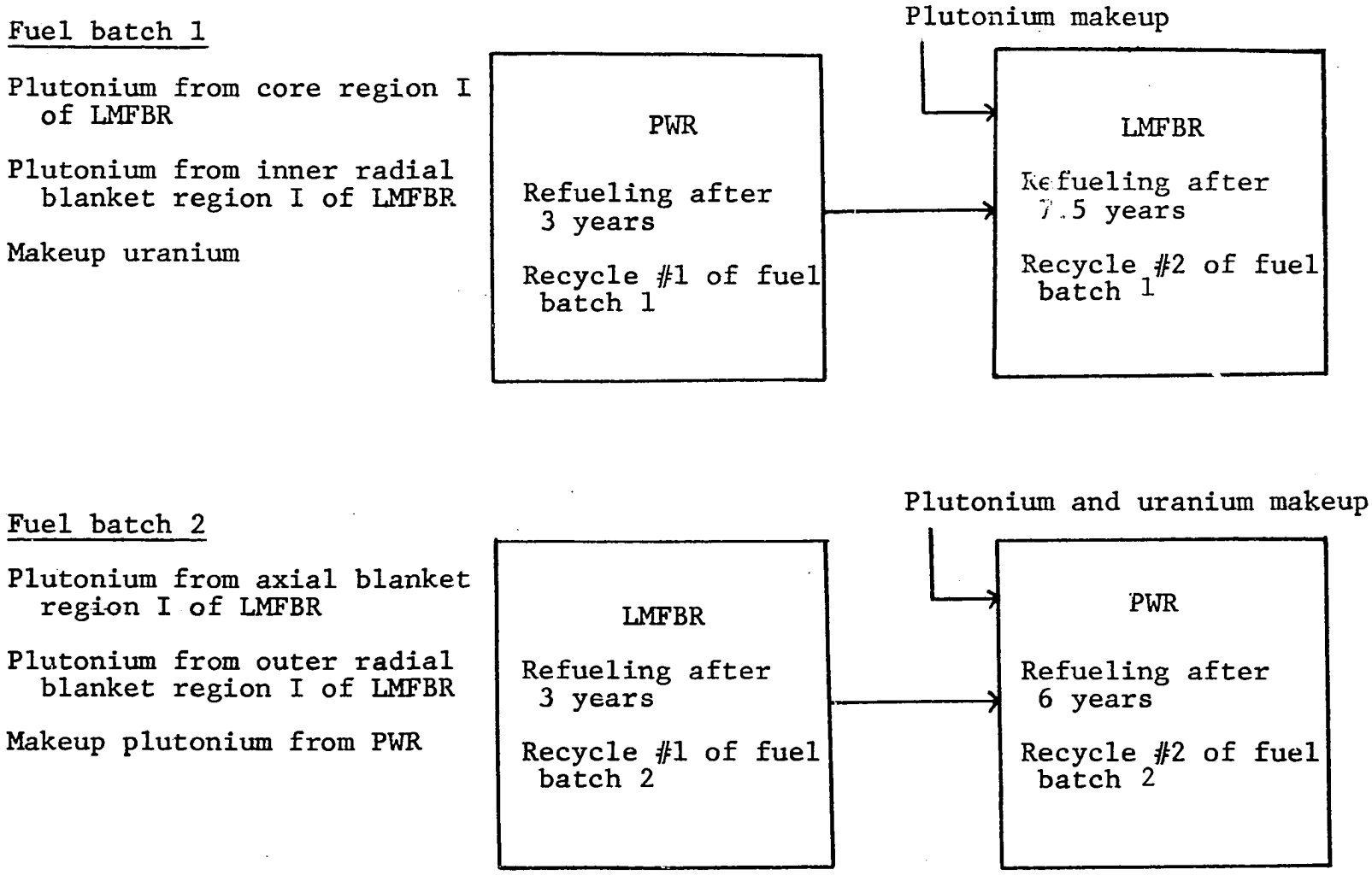

Figure 11. Nominal recycle program used to start the optimization problem

subject to

$$\delta \underline{n}_k \leq \underline{b} \qquad k = 1, 2, \ldots, K \text{ (convergence constraint)}$$

$$\underline{c}_k^T \delta \underline{n}_k = d_k \qquad k = 1, 2, \ldots, K \text{ (criticality constraint)}$$

$$\sum_k \underline{e}_k^T \delta \underline{n}_k = 0 \qquad k = 1, 2, \ldots, K \text{ (conservation of available mass constraint)} \tag{36}$$

$$\underline{f}^T \delta \underline{n}_k = \sum_m g_m \underline{f}^T \underline{n}_m \qquad k = 1, 2, \ldots, K \text{ (non-separation of plutonium isotopes)}$$

where $\underline{a}_k$, $\underline{b}$, $\underline{c}_k$, d_k, $\underline{e}_k$, $\underline{f}$ and g_m represent constant vectors and constants, and m denotes the separate batches of discharged fuel available for making up $\underline{n}_k$. Alternatively, the problem could be recast into a form using total number densities, $\underline{n}_k$, instead of $\delta \underline{n}_k$. This would be necessary for some linear programming techniques.

Solving the linear problem for the first iteration yielded the recycle program given in Figure 12. The costs of the nominal recycle program and the new recycle program are compared in Table 6. The DDP predicted costs are also presented.

It was found on the first iteration that the discharged blanket plutonium should be recycled to the PWR. The fast reactor core plutonium should replace part of the makeup plutonium from the PWR in fueling the LMFBR. Finally, the second recycle of the second fuel batch should be recycled to a fast core. The criticality constraint used for replacing the

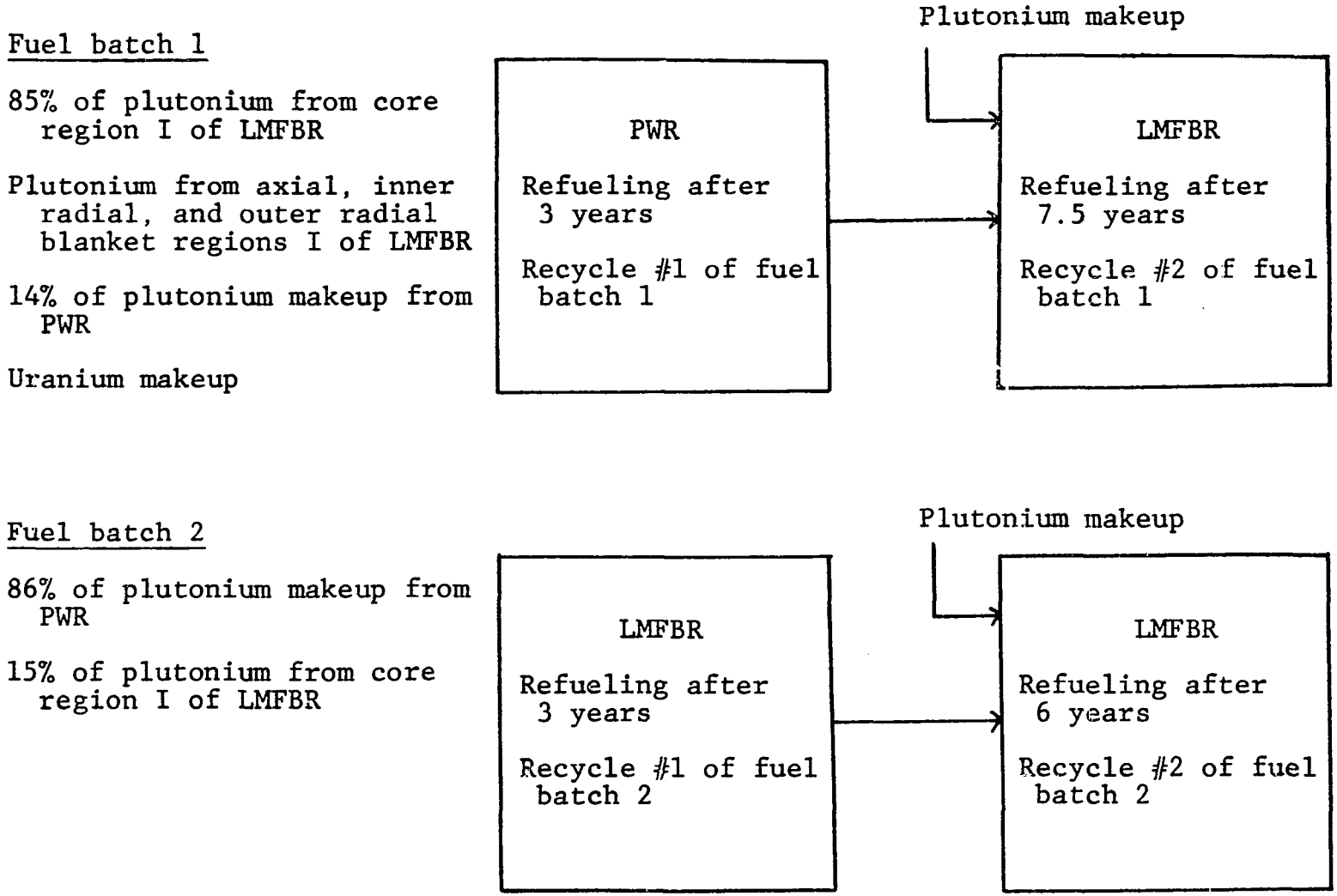

Figure 12. New nominal recycle program after the first iteration

Table 6. Comparison of costs for the initial nominal program, the DDP prediction, and the new nominal program after the first iteration

	Fuel Batch 1		Fuel Batch 2		Levelized Fuel Costs
	Recycle #1	Recycle #2[a]	Recycle #1	Recycle #2[a]	
Initial cost (mills/Kwh)	1.483	0.903	1.199	0.904	1.096
DDP predicted cost (mills/Kwh)	1.471	0.900	1.186	0.727	1.009
Actual new cost (mills/Kwh)	1.471	0.881	1.186	0.751	1.010

[a] Costs for the second recycle are present worthed to the beginning of the first recycle.

uranium-235 with fissile plutonium in the PWR was on a one-to-one basis. This approximation appeared reasonable as a constraint, as shown by the reactivity plot in Figure 13.

Repeating the procedure for a second iteration yields the results given in Figure 14 and Table 7. It might be suspected at this point that replacement of the PWR makeup plutonium used in the LMFBR core with recycle plutonium from the fast core will yield the best result. However, the accuracy of the DDP prediction would not be dependable for such a large change in number densities.

The third iteration yields the final result as presented in Figure 15 and Table 8. Hence the optimum refueling strategy for the fuel discharged from the PWR is to recycle to the PWR

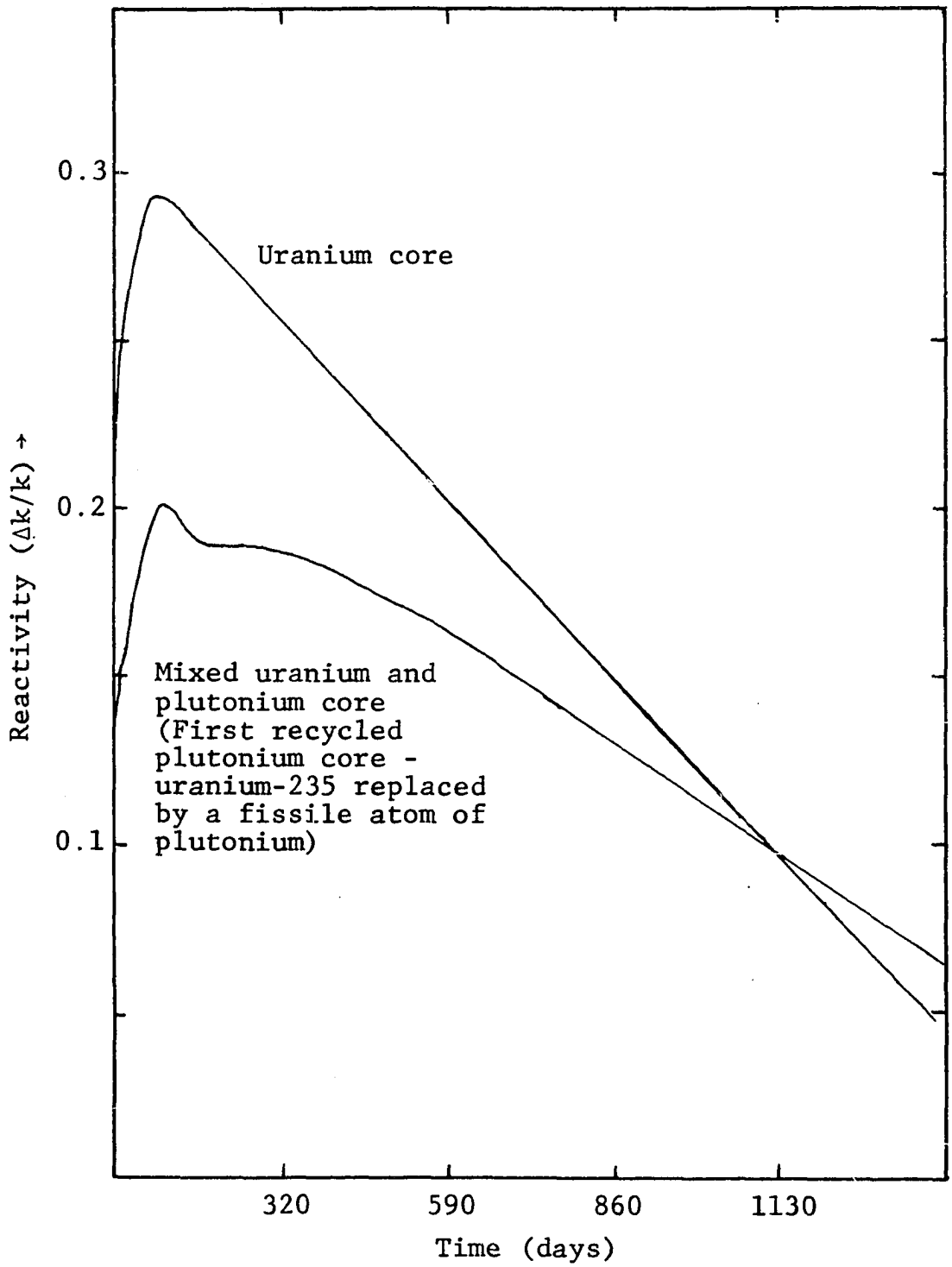

Figure 13. Comparison of reactivity for a uranium core and a mixed uranium and plutonium core as a function of operating time

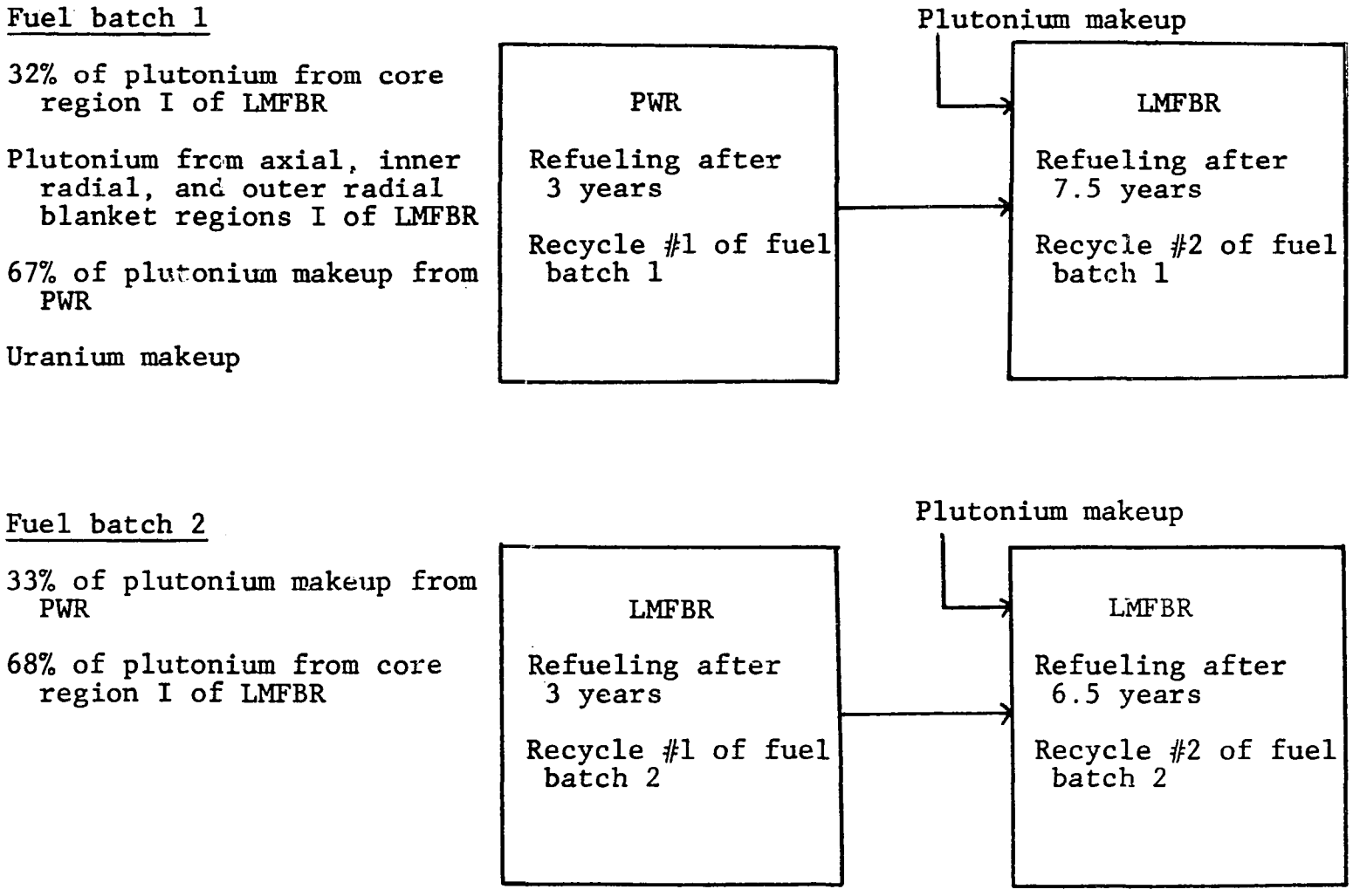

Figure 14. New nominal recycle program after the second iteration

Table 7. Comparison of costs for the initial nominal program, the DDP prediction, and the new nominal program after the second iteration

	Fuel Batch 1		Fuel Batch 2		Levelized Fuel Costs
	Recycle #1	Recycle #2[a]	Recycle #1	Recycle #2[a]	
Initial cost (mills/Kwh)	1.471	0.881	1.186	0.751	1.010
DDP predicted cost (mills/Kwh)	1.508	0.882	1.130	0.756	1.000
Actual new cost (mills/Kwh)	1.508	0.868	1.130	0.753	0.996

[a] Costs for the second recycle are present worthed to the beginning of the first recycle.

and then to the LMFBR. The blanket plutonium from the LMFBR should be recycled to the PWR and then to the LMFBR. The fuel discharged from the LMFBR core should be recycled back to the LMFBR during the next two cycles with some makeup plutonium.

To optimize the entire time period requires that all refueling strategies be optimized simultaneously. This could be done in principle using linear programming techniques. It may be achieved more readily, however, by optimizing each refueling strategy individually, and then iteratively returning to the first reloading and repeating such an optimization strategy until convergence is achieved.

If the problem considered were to be studied by dynamic

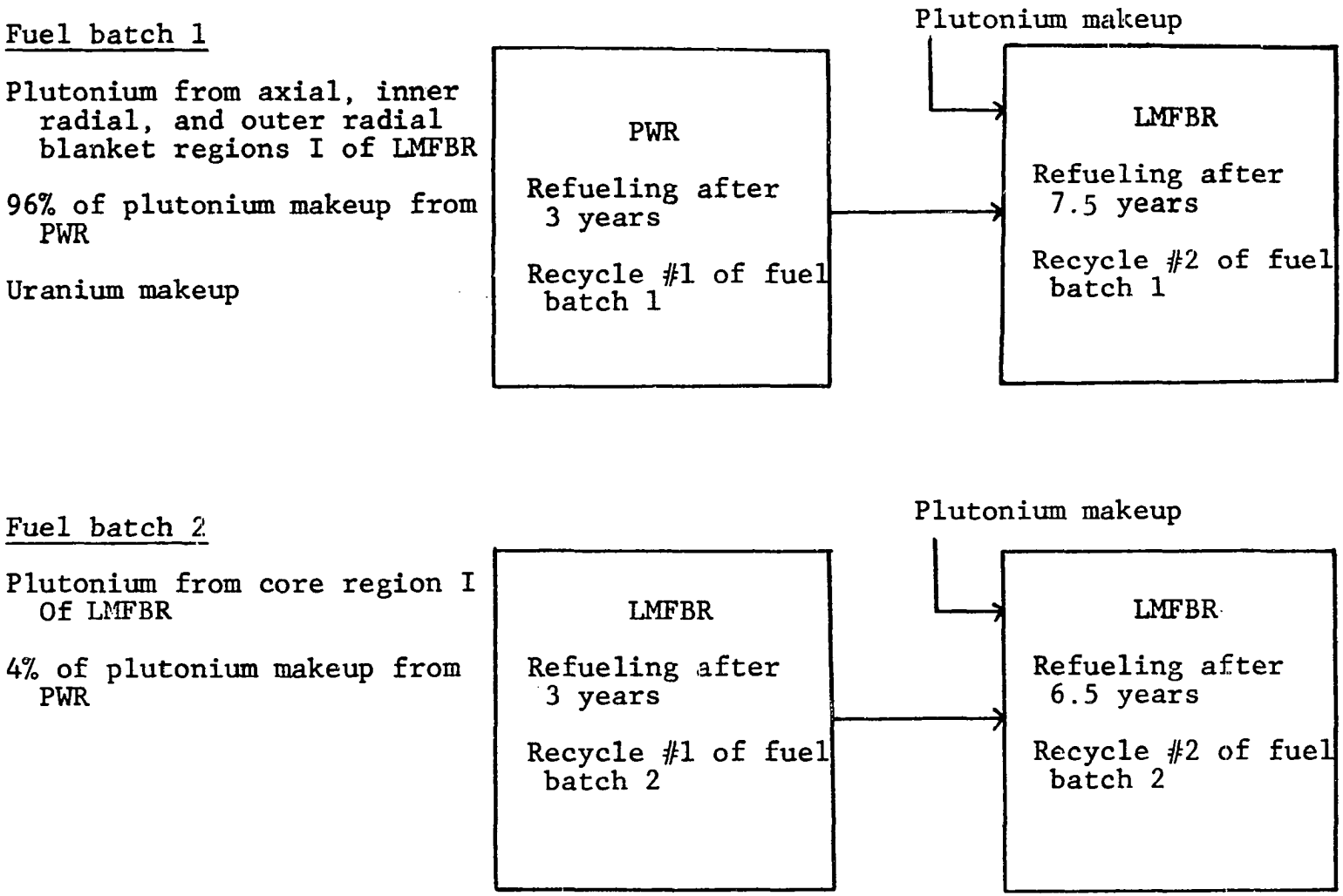

Figure 15. New nominal recycle program after the third iteration

Table 8. Comparison of costs for the initial nominal program, the DDP prediction, and the new nominal program after the third iteration

	Fuel Batch 1		Fuel Batch 2		Levelized Fuel Costs
	Recycle #1	Recycle #2[a]	Recycle #1	Recycle #2[a]	
Initial cost (mills/Kwh)	1.508	0.868	1.130	0.753	0.996
DDP predicted costs (mills/Kwh)	1.512	0.867	1.097	0.720	0.997
Actual new cost (mills/Kwh)	1.512	0.866	1.096	0.739	0.982

[a] Costs for the second recycle are present worthed to the beginning of the first recycle

programming, it is estimated that the reactor physics calculations would have to be determined a minimum of eight times. This compares with three times using DDP. The linear search might require the computational effort of one or two reactor depletion calculations. Hence, the DDP approach appears more attractive. However, no error is incurred in dynamic programming except for depletion computation errors. The DDP method introduces an error due to the truncation of the Taylor series which increases with an increasing number of stages. This is because the results of stage i+1 depend on the output of stage i. If the error in a stage is ε, the error after N stages

would be $1 - (1-\varepsilon)^N$. The error in the second reactor fuel loadings for the above calculations varied from 2 to 14%. Assuming an average of 7%, the error in determining the cost contribution of the fifth reactor fuel loadings would be almost 30%. However, the present worth factor might decrease the importance of this error when compared to errors in previous stages. Hence, the usefulness of conducting such a study beyond four or five reactor fuel loadings is limited by errors and the importance of the cost savings as compared to early cost savings.

SUMMARY AND CONCLUSIONS

The method of Differential Dynamic Programming has been modified and expanded for solving a variety of nuclear fuel management problems. The first derivatives of the state vector function were approximated as the elements of a state transition matrix. The performance index was given by the levelized fuel cycle costs. In comparing terms of the Taylor series expansion, it was found necessary only to retain the first and second terms. This yielded an iterative linear programming problem.

When applied to a batch loaded reactor, DDP gave very good cost estimates as a function of core enrichment. When applied to a multi-region reactor, the cost estimates were poor when the interaction between regions produced different flux profiles for different core design possibilities. This caused the burnups in the various regions to change, which in turn gave different costs. A procedure for estimating the effect of burnup changes was developed. This procedure had limited success and application since the buildup of other fissile isotopes and the power control problem complicate the physical model. To avoid this problem it was necessary to assume that the fuel (lattice) burnup problem can be separated from the core burnup problem, and to assume that the power control optimization problem can be separated from the economic optimization problem. The first assumption is generally valid provided the

power densities are reasonably determined, the changes in core region compositions are not too great, and there is some flexibility in the control capabilities. Since the economic optimization problem has the greatest effect on burnup, the second assumption is also good. While these assumptions may limit the optimization problem somewhat, they permit a broader application of DDP by simplifying the problem being considered while still yielding useful results.

Using these assumptions the optimum plutonium recycle loading for a core region was determined. Four iterations were required to determine this optimum.

DDP was applied to fuel shuffling with limited success. If the number densities of the isotopes for the two assemblies were not too different, an accurate cost prediction could be made. In shuffling two assemblies whose number densities were quite different, such as a fresh assembly and a partially burned assembly, the method did not give good results. Here the bounds of the neighborhood search domain were exceeded.

DDP was also used in looking at an integrated recycle program between a PWR and LMFBR. Good results were obtained for this study. The method is useful for predicting how a change in the initial core composition will effect fuel cycle costs of a core that is later loaded with recycle fuel. In this study DDP proved to be faster and required less storage requirements than would be necessary for dynamic programming.

Unfortunately, errors multiply as the number of recycle loadings increase. A large error may be expected after five cycles. This error, however, will not be as important as an error of the same magnitude at an earlier time, since the worth of money changes with time. Nevertheless, the number of recycle loadings capable of being accurately considered using DDP will be limited.

Finally, it should be noted that the DDP solution may not necessarily be a global optimum. Hence, several starting points may have to be chosen to assure a global optimum has been found.

FUTURE STUDIES

Several ideas for future study evolved during the course of this study. The reactor could be more accurately modeled by including more detail in the reactor physics calculations. More constraints on the solution may also be desirable. The detail in the model, however, should not be arbitrarily increased if there are other errors in the DDP procedure which are much larger than those from the model. The main errors will occur from the reactor model and the truncation of the Taylor series. A comparison of these errors and the relative effect of increasing the detail in the reactor model or decreasing the bounds of the search domain would be of interest.

Alternately, one could look at methods of simplifying the DDP technique further to increase its usefulness and competitiveness. One suggestion is to reduce the dimensions of the state vector.

Another possibility is to improve certain aspects of the DDP procedure while reducing the complexity of other aspects. This may also result in an overall improvement for the problem being considered. It may also be possible to approach a fuel management problem using DDP, but in an entirely different way.

One may wish to apply DDP to different reactor concepts than considered in this study. Associated with this idea would be to look at the thorium-uranium fuel cycle.

Several aspects of the DDP procedure could be examined in

more detail. The criteria for the bounds of the search domain could be explored further. This will determine to some extent the number of iterations required.

Another study could examine the functional relationships in the physics equations, such as in the burn matrix. This might result in a better approximation for the derivatives of the state vector function or for predicting burnup effects more accurately.

The DDP procedure does not guarantee a global optimum. The uncertainty in the existence of local minima and maxima in the performance index could be explored by mapping the surface of this index. The effectiveness of choosing different starting points or coupling the DDP method with another method to achieve a global capability could be studied.

It would be of interest to conduct a more detailed comparison of DDP to other methods. One could possibly develop rules when one method should be used before another. The possibility of coupling methods could also be explored. In particular DDP might be useful to refine a design determined by another procedure. For example, one might use an assignment algorithm to give shuffling patterns and then use DDP to look at small changes in the enrichment of the fresh assemblies.

Finally, it is noted that the results of this study depend on the cost parameters and cost models used. A parametric

study could be conducted or different cost models used. The cost model for plutonium would be of particular interest.

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